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PROCEEDINGS

OF THE

EDINBURGH

MATHEMATICAL SOCIETY

VOLUME XXXI.

SESSION 1912-1913.

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The following Presents to the Library have been received, for which the Society tenders its grateful thanks :—

1. American Journal of Mathematics.
2. Bulletin of the American Mathematical Society.
3. Proceedings of the American Philosophical Society.
4. Annals of Mathematics.
5. Scientific Papers, University of Sydney, 1908–12, A.
6. Proceedings of the Cambridge Philosophical Society.
7. Proceedings of the Royal Society of Edinburgh.
8. Bulletin de la Société Mathématique de France.
9. Journal de Mathématiques Eléméntaires, publié par M. H. VUIBERT.
10. Communication de la Société de Kharkow.
11. Proceedings of the London Mathematical Society.
12. Memoirs and Proceedings of the Manchester Literary and Philosophical Society.
13. The Mathematical Gazette.
14. Revista de la Real Academia de Ciencias exactas, fisicas y naturales de Madrid.
15. Annales de la Faculté des Sciences de Marseille.
16. The Journal of the Municipal School of Technology of Manchester.
17. Sitzungsberichte der Koniglich Bayerisehen Akademie der Wiesenschaften Mathematische-Physikalische Krasse München.
18. Niew Archief voor Wiskunde.
19. Proceedings and Transactions of the Nova Scotian Institute of Science.
20. Rendiconti del Circolo Matematico di Palermo.
21. Annaes Scientificos da Academia Polytechnica do Porto.
22. Revue Semestrielle des Publications Mathématique.
23. Proceedings of the Tôkyô Mathematico-Physical Society.
24. Annales de la Faculté des Sciences de la l'Université de Toulouse.
25. Nova Acta regiae Societatis Scientiarum Upsaliensis.
26. Prace Matematyczno-Fizyczne, Warsaw.
27. Wiskundige Opgaven met de Oplossingen.
28. Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich.
29. Bollettino della "Mathesis" società italiana di matematica.
30. Mémoires de la Société Royale des Sciences de Liége.
31. Journal of the Indian Mathematical Society.

32. Bulletin de la Societe Physico-Mathematique de Kasan.
33. Cambridge University Library Report.
34. Science Bulletin of the University of Kansas.
35. Mathematisch-naturwissenschaftliche Mitteilungen, Würtemburg.
36. Génesis by Arturo Soria y Mata.
37. Relazioni razionali pel Calcolo della Distanza Epicentrale; G. Negri.
38. Les Mathématiques en Portugal; R. Guimãraes.
39. Industrial Mathematics; H. W. Marsh.
40. Les propriétés cinématiques et dynamiques; L. Bachelier.
41. Professor A. P. Gruzintsev—Thirty years of his scientific, educational, and social work.

PROCEEDINGS

OF THE

EDINBURGH

MATHEMATICAL SOCIETY.

THIRTY-FIRST SESSION, 1912-1913.

For this Session the following Office-Bearers were elected:—

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On the expansion of $\left(1+\frac{z}{\lfloor 2}+\frac{z^2}{\lfloor 3}+\ldots\right)^{-n}$ in positive integral powers of z, when n is a positive integer.

By F. E. Edwardes.

(*Received 14th March 1912. Read 8th November 1912*).

§1. *The radius of convergence of the power series is* 2π.

The function $\left(1+\frac{z}{\lfloor 2}+\frac{z^2}{\lfloor 3}+\ldots\right)^{-n}$ or $\frac{z^n}{(e^z-1)^n}$ is regular within a circle whose centre is the origin of the z plane and radius 2π, and can be expanded in a Taylor's series converging at all points within the circle.

§2. *The coefficient of* z^{n-1} *in* $\left(1+\frac{z}{\lfloor 2}+\frac{z^2}{\lfloor 3}+\ldots\right)^{-n}$ *is* $(-1)^{n-1}$.

The coefficient of z^{n-1} is $\frac{1}{2\pi i}\int_C \frac{dz}{(e^z-1)^n}$, C being a closed contour surrounding the origin and lying within the circle of convergence.

Now $$-\frac{1}{n-1}\frac{d}{dz}\frac{1}{(e^z-1)^{n-1}}=\frac{e^z}{(e^z-1)^n}=\frac{1}{(e^z-1)^n}+\frac{1}{(e^z-1)^{n-1}}.$$

$$\therefore \int_C \frac{dz}{(e^z-1)^n}=-\int_C \frac{dz}{(e^z-1)^{n-1}}=(-1)^2\int_C \frac{dz}{(e^z-1)^{n-2}}=\ldots$$

$$=(-1)^{n-1}\int_C \frac{dz}{e^z-1}.$$

But the residue of $\frac{1}{e^z-1}$ is 1.

$$\therefore \int_C \frac{dz}{(e^z-1)^n}=(-1)^{n-1}\,2\pi i.$$

Hence the coefficient of z^{n-1} is $(-1)^{n-1}$.

§3. *If the coefficient of z^r in* $\left(1+\frac{z}{\lfloor 2}+\frac{z^2}{\lfloor 3}+\ldots\right)^{-n}$ *is denoted by* ${}_na_r$, *then*

$$ {}_na_r = -{}_{n-1}a_{r-1} + \frac{n-r-1}{n-1}\,{}_{n-1}a_r, $$

n having any of the values 2, 3, 4..., *and r any of the values* 1, 2, 3... .

For $${}_{n-1}a_{r-1} = \frac{1}{2\pi i}\int \frac{z^{n-r-1}}{(e^z-1)^{n-1}}\,dz = \frac{1}{2\pi i}\int \frac{z^{n-r-1}e^z - z^{n-r-1}}{(e^z-1)^n}dz$$

$$= -\frac{1}{2\pi i}\int \frac{z^{n-r-1}}{(e^z-1)^n}dz - \frac{1}{2\pi i}\frac{1}{n-1}\int z^{n-r-1}\frac{d}{dz}\frac{1}{(e^z-1)^{n-1}}dz$$

$$= -{}_na_r + \frac{1}{2\pi i}\frac{n-r-1}{n-1}\int_C \frac{z^{n-r-2}}{(e^z-1)^{n-1}}dz$$

$$= -{}_na_r + \frac{n-r-1}{n-1}\,{}_{n-1}a_r.$$

Hence ${}_na_r = -{}_{n-1}a_{r-1} + \frac{n-r-1}{n-1}\,{}_{n-1}a_r$, provided $n>1$, $r>0$.

It may be added that since the coefficient of z^0 is unity for $n=1, 2, 3, \ldots$ we have

$${}_na_0 = {}_{n-1}a_0 = \ldots = {}_2a_0 = {}_1a_0.$$

§4. The values of the suffixes at which we ultimately arrive are seen very clearly if we associate with each coefficient ${}_na_r$ a point whose ordinate is n and abscissa r: this point can be denoted by the same symbol ${}_na_r$ without any danger of confusion. Provided r is not equal to $n-1$, the reduction equation leads from the point ${}_na_r$ to the two points ${}_{n-1}a_r$ and ${}_{n-1}a_{r-1}$, that is, to those reached by a step one unit in length parallel to the axis of n and towards the axis of r, and then another parallel to the axis of r and towards the axis of n. If $r>$ or $=n$, the repeated application of the reduction equation will therefore determine ${}_na_r$ as the sum of multiples of ${}_1a_r$, ${}_1a_{r-1}$, $\ldots$ ${}_1a_{r-n+1}$, that is, as the sum of multiples of

certain of Bernoulli's numbers, since when $n=1$ the expansion becomes

$$1-\frac{z}{2}+\frac{B_1}{\lfloor 2}z^2-\frac{B_2}{\lfloor 4}z^4+\dots .$$

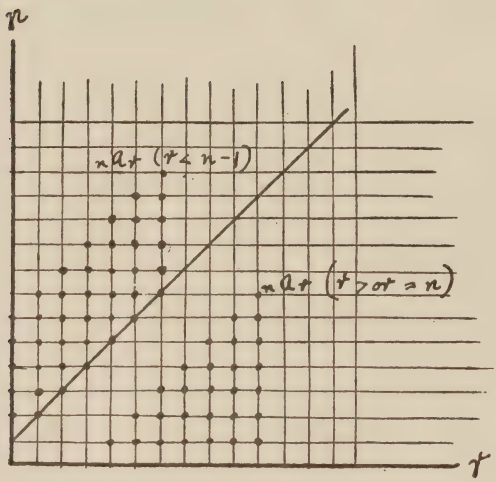

Fig. 1.

For powers of z lower than z^{n-1} the case is different. When the suffix r is less than the suffix n by unity, the reduction equation assumes the form

$${}_ka_{k-1}=-{}_{k-1}a_{k-2},$$

and again when the suffix r is zero,

$${}_ka_0={}_{k-1}a_0.$$

These discontinuities in the form of the reduction equation preclude the crossing of the lines $r=n-1$ and $r=0$, and lead to the determination of ${}_na_r$ when $r<n-1$, simply as a multiple of ${}_1a_0$.

§5. *The coefficients of powers of z higher than z^{n-1} in the expansion of* $\left(1+\frac{z}{\lfloor 2}+\frac{z^2}{\lfloor 3}+\dots\right)^{-n}$ *are given, for values of* $r>n-1$,

by

$$ {}_nc_r = \frac{(-1)^{n-1}}{\underline{|n-1}} (\delta+1)(2\delta+1)\ldots(\overline{n-1}\,\delta+1)_1 c_r, $$

where ${}_nc_r \equiv \underline{|r-n}\,{}_na_r$, ${}_kc_k = {}_ka_k$, *and* δ *is a symbolic operator such that* $\delta({}_mc_k) = {}_mc_{k-1}$.

When $r > n-1$, the reduction equation

$$ {}_na_r = -{}_{n-1}a_{r-1} - \frac{r-n+1}{n-1}{}_{n-1}a_r $$

may be written

$$ \underline{|r-n}\,{}_na_r = -\underline{|r-n}\,{}_{n-1}a_{r-1} - \frac{1}{n-1}\,\underline{|r-\overline{n-1}}\,{}_{n-1}a_r, $$

or

$$ {}_nc_r = -{}_{n-1}c_{r-1} - \frac{1}{n-1}\,{}_{n-1}c_r, $$

where ${}_nc_r \equiv \underline{|r-n}\,{}_na_r$ when $r > n$, and ${}_kc_k \equiv {}_ka_k$.

This modified reduction equation holds good for all values of $n > 1$.

Now let Δ and δ be symbolic operators such that

$$ \Delta({}_mc_k) = {}_{m-1}c_k \text{ and } \delta({}_mc_k) = {}_mc_{k-1}. $$

Then

$$ {}_nc_r = -\left(\delta + \frac{1}{n-1}\right)\Delta\,{}_nc_r, $$

so that the operator $-\left(\delta+\frac{1}{n-1}\right)\Delta$ acting on a coefficient whose primary suffix in n reproduces that coefficient: and $\delta\Delta\,{}_nc_r$ and $\Delta\,{}_nc_r$ being coefficients whose primary suffix is $n-1$, and the operators being commutative, it follows that

$$ {}_nc_r = (-1)^2\left(\delta+\frac{1}{n-1}\right)\left(\delta+\frac{1}{n-2}\right)\Delta^2{}_nc_r. $$

Thus

$$ {}_nc_r = (-1)^{n-1}\left(\delta+\frac{1}{n-1}\right)\left(\delta+\frac{1}{n-2}\right)\ldots(\delta+1)\,\Delta^{n-1}{}_nc_r. $$

$$ = \frac{(-1)^{n-1}}{\underline{|n-1}}(\delta+1)(2\delta+1)\ldots(\overline{n-1}\,\delta+1)\,{}_1c_r. $$

This formula gives the coefficient ${}_na_r$ for values of $r > n-1$ as the sum of multiples of certain numbers of Bernoulli.

§ 6. *For values of* r *from* $n-2$ to 1

$$ {}_na_r = (-1)^r \frac{\lfloor n-r-1}{\lfloor n-1} \, S_r(1, 2, 3 \ldots n-1), $$

where $S_r(1, 2, 3 \ldots n-1)$ *means the sum of the products of the numbers* $1, 2, 3 \ldots n-1$ *taken* r *at a time.*

When $0 < r < n-1$ the reduction equation may be written

$$ \frac{{}_na_r}{\lfloor n-r-1} = -\frac{{}_{n-1}a_{r-1}}{\lfloor n-r-1} + \frac{1}{n-1} \frac{{}_{n-1}a_r}{\lfloor n-r-2}, $$

or

$$ {}_nb_r = -{}_{n-1}b_{r-1} + \frac{1}{n-1} {}_{n-1}b_r, $$

where

$$ {}_nb_r \equiv \frac{{}_na_r}{\lfloor n-r-1}. $$

When, however, $r = 0$, and again when $r = n-1$, the reduction equation is discontinuous in form. In the first case

$$ {}_ka_0 = {}_{k-1}a_0 \quad (k > 1), $$

and in the second case

$$ {}_ka_{k-1} = -{}_{k-1}a_{k-2} \quad (k > 1); $$

and the corresponding modified equations are

$$ {}_kb_0 = \frac{1}{k-1} {}_{k-1}b_0 \quad (k > 1), $$

and

$$ {}_kb_{k-1} = -{}_{k-1}b_{k-2} \quad (k > 1). $$

Hence the modified equation may be regarded as continuous in form if we introduce the symbols

$${}_kb_{-1}, {}_kb_{-2}, \ldots \text{ etc.}, {}_{k-1}b_{k-1}, {}_{k-1}b_{k-2}, \ldots \text{ etc.},$$

all of them having the value zero. By the introduction of these

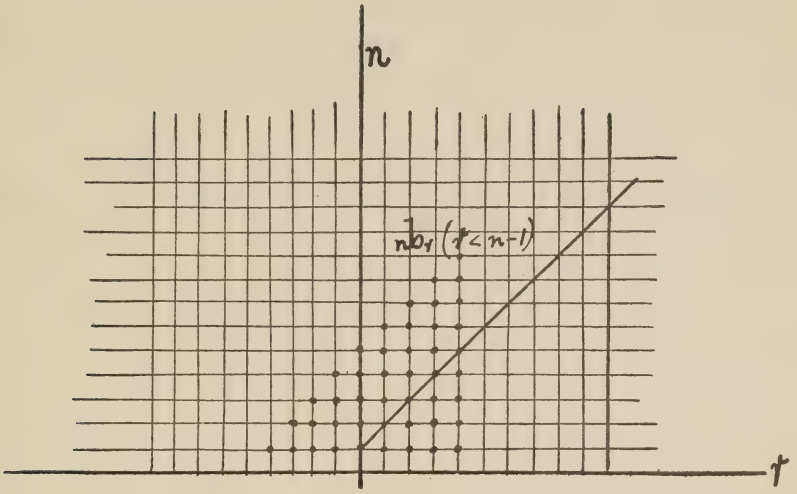

Fig. 2.

symbols, the array of points associated with the coefficients obtained from ${}_nb_r$ by means of the reduction equation is no longer bounded by the lines $r=0$, $r=n-1$, but extends to the line $n=1$. Of the points ${}_1b_r$, ${}_1b_{r-1}$, ... ${}_1b_1$, ${}_1b_0$, ${}_1b_{-1}$, ..., etc., ultimately reached, all except ${}_1b_0$ are associated with coefficients which vanish by virtue of their definition.

Now let Δ and δ be symbolic operators such that $\Delta({}_mb_k)={}_{m-1}b_k$ and $\delta({}_mb_k)={}_mb_{k-1}$.

Then
$${}_nb_r=\left(-\delta+\frac{1}{n-1}\right)\Delta\,{}_nb_r=\left(-\delta+\frac{1}{n-1}\right)\left(-\delta+\frac{1}{n-2}\right)\Delta^2\,{}_nb_r=\ldots$$

$$=\left(-\delta+\frac{1}{n-1}\right)\left(-\delta+\frac{1}{n-2}\right)\ldots(-\delta+1)\,\Delta^{n-1}\,{}_nb_r$$

$$=\left(-\delta+\frac{1}{n-1}\right)\left(-\delta+\frac{1}{n-2}\right)\ldots(-\delta+1)\,{}_1b_r$$

$$=\frac{1}{\underline{|\,n-1}}\,(1-\delta)(1-2\delta)\ldots(1-\overline{n-1}\delta)\,{}_1b_r.$$

But $\delta^{\lambda}({}_1b_r) = 0$, for all values of λ except r, and $\delta^{r}({}_1b_r) = 1$.

$$\therefore \quad {}_nb_r = \frac{(-1)^n}{\lfloor n-1} \, S_r(1, 2, 3 \ldots n-1),$$

where $S_r(1, 2, 3 \ldots n-1)$ means the sum of the products of the numbers $1, 2, 3 \ldots n-1$ taken r at a time.

$$\text{Hence} \quad {}_na_r = (-1)^r \frac{\lfloor n-r-1}{\lfloor n-1} \, S_r(1, 2, 3, \ldots n-1)$$

for all values of $r < n-1$.

It may be observed that the formula holds for $r = n-1$, but the coefficients for which this is the case have been evaluated already.

§7. A number of identities involving Bernoulli's numbers can be obtained by expanding $\left(1 - \frac{z}{2} + \frac{B_1}{\lfloor 2} z^2 - \frac{B_2}{\lfloor 4} z^4 + \ldots\right)^n$ and equating the coefficients of various powers of z to those of the expansion of $\left(1 + \frac{z}{\lfloor 2} + \frac{z^2}{\lfloor 3} + \ldots\right)^{-n}$. For example, when $n = 2$ the coefficients of z^{2r} give

$$(2r+1)\frac{B_r}{\lfloor 2r} = \frac{B_{r-1}}{\lfloor 2r-2}\frac{B_1}{\lfloor 2} + \frac{B_{r-2}}{\lfloor 2r-4}\frac{B_2}{\lfloor 4} + \ldots + \frac{B_1}{\lfloor 2}\frac{B_{r-1}}{\lfloor 2r-2}.$$

On the Summation of $1^r + 2^r + 3^r + \ldots + n^r$.

By J. A. Donaldson.

(*Read 14th February 1913. Received 18th February 1913.*)

We shall use the notation $\Sigma n^r \equiv 1^r + 2^r + \ldots + n^r$. Mr A. J. Gray suggested to me that $n(n+1)$ is always a factor of Σn^r, and that, in addition, $2n+1$ is a factor when r is even.

§1. That $n(n+1)$ is a factor of Σn^r is proved in Chrystal's Algebra I., chap. XX., §9, but the following proof seems somewhat simpler:—

Let $\Sigma n^r = f(n)$.

We know that $f(n)$ is an integral algebraic function of n.

When $n=0$, $\Sigma n^r = 0$; *i.e.* $f(n)=0$; therefore $f(n)$ contains the factor n.

Hence $f(n+1)$ contains the factor $(n+1)$.

Now $\Sigma(n+1)^r = f(n) + (n+1)^r = f(n+1)$,

and $f(n+1)$ and $(n+1)^r$ both contain the factor $(n+1)$;

$\therefore$ $f(n)$ contains the factor $(n+1)$;

i.e. Σn^r contains the factor $n(n+1)$.

§2. To prove that $(2n+1)$ is a factor when r is even, we must attempt to calculate the even sums without making use of the odd ones. In the identity

$$(x+1)^r - (x-1)^r \equiv 2rx^{r-1} + {}_rC_3 x^{r-3} + \ldots + {}_rC_2 x^2 + 2,$$

which holds *when r is odd*, put successively $x=n$, $x=n-1$, $\ldots x=2$, $x=1$, and we have

$$\begin{aligned}
(n+1)^r - (n-1)^r &= 2rn^{r-1} + \ldots\ldots\ldots + {}_rC_2 n^2 + 2 \\
n^r - (n-2)^r &= 2r(n-1)^{r-1} + \ldots + {}_rC_2 (n-1)^2 + 2 \\
(n-1)^r - (n-3)^r &= 2r(n-2)^{r-1} + \ldots + {}_rC_2 (n-2)^2 + 2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
3^r - 1^r &= 2r2^{r-1} + \ldots\ldots\ldots\ldots + {}_rC_2 2^2 + 2 \\
2^r - 0 &= 2r1^{r-1} + \ldots\ldots\ldots\ldots + {}_rC_2 1^2 + 2.
\end{aligned}$$

If we add these equations together, we have

$$(n+1)^r + n^r - 1 = 2r\Sigma n^{r-1} + \ldots + 2_rC_2\Sigma n^2 + 2n\,;$$

i.e. $$(n+1)^r + n^r - (2n+1) = 2r\Sigma n^{r-1} + \ldots + 2_rC_2\Sigma n^2.$$

When we put $n = -\frac{1}{2}$, the left-hand side of this last equation vanishes; hence, since Σn^2 is divisible by $2n+1$, we can prove successively that Σn^4, Σn^6, .. Σn^{2s} are all divisible by $2n+1$.

§3. When examining the above, I worked out the sums as far as Σn^7 and factorised them. The expressions for Σn^3, Σn^5, and Σn^7 contained the factor $n^2(n+1)^2$ and suggested $n^2(n+1)^2$ as a factor of Σn^r when r is odd, except in the case of $r = 1$.

To prove this we have the identity

$$(x+1)^r - (x-1)^r \equiv 2rx^{r-1} + 2_rC_3x^{r-3} + \ldots + 2_rC_3x^3 + 2rx,$$

which holds *when r is even.*

When we give x the values n, $n-1$, ... 2, 1, and add the resulting identities, we have

$$(n+1)^r + n^r - 1 = 2r\Sigma n^{r-1} + \ldots\ldots + 2_rC_3\Sigma n^3 + 2r\Sigma n\,;$$

i.e. $$(n+1)^r + n^r - 1 - rn(n+1) = 2r\Sigma n^{r-1} + \ldots + 2_rC_3\Sigma n^3.$$

When we expand the left-hand side in powers of n, the terms below n^2 are absent; therefore n^2 is a factor of the left-hand side. Also when we put $n+1 = m$ in the left side and expand in powers of m, the terms below m^2 are absent; *i.e.* m^2 or $(n+1)^2$ is a factor of the left side.

Now $n^2(n+1)^2$ is a factor of Σn^3; hence we can show successively that it is a factor of Σn^5, Σn^7 .. Σn^{2s+1}.

Hence we have proved that:—

Σn^r *contains the factor* $n(n+1)(2n+1)$ *when* r *is even,* and *the factor* $n^2(n+1)^2$ *when* r *is odd and greater than* 1.

§4. *Expression of* Σn^r *in powers of* n.

From the section in Chrystal's Algebra, referred to above, we learn that Σn^r is an integral function of n of the $(r+1)^{\text{th}}$ degree, say

$$\Sigma n^r = {}_ra_0n^{r+1} + {}_ra_1n^r + {}_ra_2n^{r-1} + \ldots + {}_ra_{r-1}n^2 + {}_ra_rn. \qquad \text{I.}$$

Hence we have

$${}_ra_0n^{r+1} + {}_ra_1n^r + \ldots + {}_ra_rn + (n+1)^r = \Sigma(n+1)^r$$
$$= {}_ra_0(n+1)^{r+1} + {}_ra_1(n+1)^r + \ldots + {}_ra_r(n+1). \qquad \text{II.}$$

II. is an identity and true for all positive integral values of n, and so we may equate coefficients of like powers of n giving :—

n^r $\quad {}_ra_0 \times {}_{r+1}C_1 = 1$, ...(1)

n^{r-1} $\quad {}_ra_0 \times {}_{r+1}C_2 + {}_ra_1 \times {}_rC_1 = {}_rC_1$, ...(2)

n^{r-2} $\quad {}_ra_0 \times {}_{r+1}C_3 + {}_ra_1 \times {}_rC_2 + {}_ra_2 \times {}_{r-1}C_1 = {}_rC_2$, ...(3)

.....

n^{r-s} $\quad {}_ra_0 \times {}_{r+1}C_{s+1} + {}_ra_1 \times {}_rC_s + \ldots + {}_ra_s \times {}_{r-s+1}C_1 = {}_rC_s$, ... $(s+1)$

.....

n $\quad (r+1){}_ra_0 + r{}_ra_1 + (r-1){}_ra_2 + \ldots + 2{}_ra_{r-1} = r$, ...$(r)$

constant ${}_ra_0 + {}_ra_1 + {}_ra_2 + \ldots + {}_ra_{r-1} + {}_ra_r = 1$. ....$(r+1)$

The $(r+1)^{\text{th}}$ equation is also the expression of the fact that the identity I. holds when $n = 1$.

These equations enable us to calculate ${}_ra_0$, ${}_ra_1$, ..., ${}_ra_r$ successively in terms of r.

Evidently ${}_ra_{r+1}$ is always equal to zero, since Σn^r is always divisible by n. We shall see that the equations (1)...$(r+1)$ give us no information as to ${}_ra_{r+1}$.

From (1), ${}_ra_0 = \dfrac{1}{r+1}$.

From (2), $\dfrac{1}{r+1} \quad \dfrac{(r+1)r}{2\,!} + r{}_ra_1 = r$,

giving $\qquad {}_ra_1 = \frac{1}{2}$.

From (3), $\dfrac{1}{r+1} \dfrac{(r+1)r(r-1)}{3\,!} + \frac{1}{2}\dfrac{r(r-1)}{2\,!} + (r-1){}_ra_2 = \dfrac{r(r-1)}{2\,!}$,

which, *except when* $r = 1$, gives ${}_ra_2 = \frac{1}{6}\dfrac{r}{2\,!}$. When $r = 1$, ${}_ra_2$ is the term ${}_ra_{r+1}$, and this shows how these equations give us no information about ${}_ra_{r+1}$.

From (4), $\dfrac{1}{r+1} \dfrac{(r+1)\ldots(r-2)}{4\,!} + \frac{1}{2}\dfrac{r\ldots(r-2)}{3\,!} + \dfrac{r}{12} \dfrac{(r-1)(r-2)}{2\,!}$

$$+ (r-2){}_ra_3 = \frac{r(r-1)(r-2)}{3\,!},$$

which reduces to ${}_ra_3 = 0 \times \dfrac{r(r-1)}{3\,!}$.

From (5), $\frac{1}{r+1}\frac{(r+1)\ldots(r-3)}{5\,!}+\frac{1}{2}\frac{r\ldots(r-3)}{4\,!}+\frac{r}{12}\frac{(r-1)\ldots(r-3)}{3\,!}$

$$+(r-3)_r a_4=\frac{r\ldots(r-3)}{4\,!},$$

which, *except when* $r=3$, gives

$$_r a_4=-\tfrac{1}{30}\frac{r(r-1)(r-2)}{4\,!}.$$

From the above sample of calculation, and the form of equations, $(1)\ldots(r+1)$, it is obvious that $_r a_s$ is of the form

$$a_s\times\frac{r(r-1)(r-2)\ldots(r-s+2)}{s\,!}$$

where the constant a_s depends on s only.

If we go back now to equation I., when $r=3$, $_3a_3=0$ since Σn^3 contains the factor n^2.

Now $_r a_3=a_3\times\frac{r(r-1)}{3\,!}$, and when $r=3$, $\frac{r(r-1)}{3\,!}$ does not vanish; therefore $a_3=0$, as we saw above by calculation.

For the same reason, viz., that Σn^r is divisible by n^2 when r is odd and greater than 1, $a_5=a_7=a_9=\ldots=0$.

Values of a are given below up to a_{18}.

s	a_s
2	1/6
4	− 1/30
6	1/42
8	− 1/30
10	5/66
12	− 691/2730
14	7/6
16	− 3617/510
18	43867/798

The values of a_s given above were not all calculated from the

equations (1)...$(r+1)$ as before, but by giving r a particular value in a way which the following example will make clear :—

$$\Sigma n^9 = {}_9a_0 n^{10} + {}_9a_1 n^9 + {}_9a_2 n^8 + {}_9a_4 n^6 + {}_9a_6 n^4 + {}_9a_8 n^2. \qquad \text{III.}$$

Our object is to calculate a_8, knowing the others up to a_6.

Equation $(r+1)$ shows, when we put $r=9$, that

$${}_9a_0 + {}_9a_1 + {}_9a_2 + {}_9a_4 + {}_9a_6 + {}_9a_8 = 1.$$

Now $\quad {}_9a_0 + {}_9a_1 + \ldots + {}_9a_6 = \frac{1}{10} + \frac{1}{2} + \frac{3}{4} - \frac{7}{10} + \frac{1}{2} = \frac{23}{20}$;

$$\therefore \quad {}_9a_8 = -\tfrac{3}{20}.$$

Also $\quad {}_ra_8 = a_8 \dfrac{r(r-1)\ldots(r-6)}{8!}$;

$$\therefore \quad {}_9a_8 = a_8 \frac{9.8\ldots3}{8!} = -\tfrac{3}{20} ;$$

whence $\quad a_8 = -\tfrac{1}{30}.$

We might also get a_8 by starting with Σn^8 for, putting $r=8$,

$$\Sigma n^8 = {}_8a_0 n^9 + {}_8a_1 n^8 + {}_8a_2 n^7 + {}_8a_4 n^5 + {}_8a_6 n^3 + {}_8a_8 n,$$

and

$${}_8a_0 + {}_8a_1 + {}_8a_2 + {}_8a_4 + {}_8a_6 + {}_8a_8 = 1.$$

In general to get a_{2s}, make r equal to $2s$ or $2s+1$, and it sometimes happens that one value will give simpler calculations than the other.

§ 5. As we have not used the fact that Σn^r is divisible by $(n+1)$ in §4, we may use these coefficients to show that Σn^r is divisible by $(n+1)$ always, and by $(n+1)^2$ when r is odd and greater than 1.

Leaving out left-hand suffixes,

$$\Sigma n^r = a_0 n^{r+1} + a_1 n^r + a_2 n^{r-1} + a_4 n^{r-3} + \ldots + a_r n \qquad (r \text{ even}),$$

$$\Sigma n^r = a_0 n^{r+1} + a_1 n^r + a_2 n^{r-1} + a_4 n^{r-3} + \ldots + a_{r-1} n^2 \quad (r \text{ odd}).$$

(1) *r even.* By $(r+1)$, $\quad a_0 + a_1 + a_2 + \ldots a_r = 1$;

$$\therefore \quad a_0 + a_2 + \ldots a_r = 1 - a_1 = \tfrac{1}{2}.$$

Now $\underset{n=-1}{\mathrm{L}} \; \Sigma n^r = a_1 - (a_0 + a_2 + \ldots a_r) = 0$;

$\therefore$ Σn^r is divisible by $(n+1)$ when r is even.

(2) r *odd.* Just as in the case of r even, we show that Σn^r is divisible by $n+1$.

Equation (r) is

$$(r+1)a_0 + ra_1 + (r-1)a_2 + \ldots 2a_{r-1} = r\,;$$

i.e. $$(r+1)a_0 + (r-1)a_2 + \ldots + 2a_{r-1} = r - a_1 r = \tfrac{1}{2}r\,;$$

i.e. $$(r+1)a_0 - ra_1 + (r-1)a_2 + \ldots + 2a_{r-1} = 0\,;$$

i.e. $$\mathrm{L}_{n=-1}\left(\frac{d}{dn}\Sigma n^r\right) = 0 \quad (r \text{ odd})\,;$$

i.e. Σn^r contains the factor $(n+1)^2$ when r is odd and greater than 1 (for equation (r) does not involve a_1 when $r=1$).

§ 6. *Proof of Bernoulli's Theorem.*

I had reached this stage when Professor Whittaker pointed out to me Bernoulli's expansion (see Chrystal's Algebra II., §7, Chap. XXVIII.). We can demonstrate this expansion from the above as follows :—

The tabulated values of a_s are simply Bernoulli's numbers with alternating signs.

From what we have proved in § 4, we can assume

$$\Sigma n^r = \frac{n^{r+1}}{r+1} + \tfrac{1}{2}n^r + \frac{\beta_1}{2}rn^{r-1} - \frac{\beta_2}{4}{}_r\mathrm{C}_3 n^{r-3} + \frac{\beta_3}{6}{}_r\mathrm{C}_5 n^{r-5} - \frac{\beta_4}{8}{}_r\mathrm{C}_7 n^{r-7} \ldots\ldots\ldots$$

where $\beta_s = |a_{2s}|$;

i.e. $$a_{2s} = (-)^{s-1}\frac{\beta_s}{2s}{}_r\mathrm{C}_{2s-1}.$$

When we substitute for the a's in $(r+1)$, we have

(1) when r is even, $= 2p$ say,

$${}_ra_0 + {}_ra_1 + {}_ra_2 + {}_ra_4 + \ldots + {}_ra_r = 1\,;$$

i.e. $$\frac{1}{r+1} + \tfrac{1}{2} + \frac{\beta_1}{2}{}_r\mathrm{C}_1 - \frac{\beta_2}{4}{}_r\mathrm{C}_3 + \ldots = 1\,;$$

i.e. $${}_{2p}\mathrm{C}_1\frac{\beta_1}{2} - {}_{2p}\mathrm{C}_3\frac{\beta_2}{4} + \ldots + (-)^{p-1}{}_{2p}\mathrm{C}_{2p-1}\frac{\beta_p}{2p} = \tfrac{1}{2} - \frac{1}{2p+1} = \frac{2p-1}{2(2p+1)}.$$

(2) When r is odd, $= 2p+1$ say,

$${}_rA_0 + {}_rA_1 + {}_rA_2 + \ldots + {}_rA_{r-1} = 1 ;$$

i.e. $$\frac{1}{r+1} + \tfrac{1}{2} + \frac{\beta_1}{2}\,{}_rC_1 - \frac{\beta_2}{4}\,{}_rC_3 + \ldots = 1 ;$$

i.e. $${}_{2p+1}C_1\frac{\beta_1}{2} - {}_{2p+1}C_3\frac{\beta_2}{4} + \ldots + (-)^{p-1}\,{}_{2p+1}C_{2p-1}\frac{\beta_p}{2p}$$

$$= \tfrac{1}{2} - \frac{1}{2p+2} = \frac{p}{2p+2}. \qquad \text{V.}$$

When we multiply IV. by $(-)^{p-1}(2p+1)$, we get, since ${}_{2p}C_{2s-1} \times \dfrac{2p+1}{2s} = {}_{2p+1}C_{2s}$,

$${}_{2p+1}C_{2p}\,\beta_p - {}_{2p+1}C_{2p-2}\,\beta_{p-1} + \ldots + (-)^{p-1}\,{}_{2p+1}C_2\beta_1 = (-)^{p-1}(p - \tfrac{1}{2}).$$
IV.^{1}

Similarly, when we multiply V. by $(-)^{p-1}(2p+2)$, we get

$${}_{2p+2}C_{2p}\,\beta_p - {}_{2p+2}C_{2p-2}\,\beta_{p-1} + \ldots + (-)^{p-1}\,{}_{2p+2}C_2\,\beta_1 = (-)^{p-1}p. \qquad \text{V.}^1$$

We can calculate the β's from IV.^{1} or V.^{1} by giving p the values 1, 2, 3... successively. *Cf.* Calculation of a_8, in end of § 4, by putting $r = 8$ or 9 in equation $(r+1)$.

These alternative recurrence formulae for β are the same as those for Bernoulli's numbers (see Chrystal's Algebra II., Equations (10′) and (11′), § 6, Chap. XXVIII.).

Hence the numbers β_1, β_2, etc., are Bernoulli's numbers B_1, B_2, etc.

Hence

$$\Sigma n^r = \frac{n^{r+1}}{r+1} + \tfrac{1}{2}n^r + \frac{B_1}{2}rn^{r-1} - \frac{B_2}{4}\,{}_rC_3\,n^{r-3} + \frac{B_3}{6}\,{}_rC_5\,n^{r-5} - \frac{B_4}{8}\,{}_rC_7\,n^{r-7} + \ldots\ldots,$$

the last term being $(-)^{\frac{1}{2}(r-2)}B_{\frac{1}{2}r}n$ or $\frac{1}{2}(-)^{\frac{1}{2}(r-3)}r\,B_{\frac{1}{2}(r-1)}n^2$ according as r is even or odd (Bernoulli's Theorem).

The proof seems rather complicated, as I have taken trouble to make every step clear, but it depends on very simple principles, and does not involve even an infinite series.

§ 7. To show how intimately connected the simple factors of Σn^r are with the Bernoullian expansions given in that section of

Chrystal referred to in § 6 of this paper, we will give a formula for calculating the Bernoullian numbers, deduced from the fact that $2n+1$ is a factor of Σn^{2p}.

Since Σn^{2p} contains the factor $2n+1$, we have, putting $n=-\frac{1}{2}$ in the expression for Σn^{2p}.

$$\frac{1}{2p+1}-1+2^2\frac{B_1}{2}2p \quad 2^4\frac{B_2}{4}{}_{2p}C_3+\ldots+(-)^{p-1}2^{2p}\frac{B_p}{2p}{}_{2p}C_{2p-1}=0.$$

i.e. taking $\dfrac{1}{2p+1}-1$ to the right-hand side of the equation and multiplying up by $2p+1$,

$$2^2B_1{}_{2p+1}C_2-2^4B_2{}_{2p+1}C_4+2^6B_3{}_{2p+1}C_6\ldots$$
$$+(-)^{p-1}2^{2p}B_p{}_{2p+1}C_{2p}=2p. \qquad \text{VI.}$$

Now $$x\frac{e^x+e^{-x}}{e^x-e^{-x}}=1+\frac{B_1}{2!}2^2x^2-\frac{B_2}{4!}2^4x^4+\frac{B_3}{6!}2^6x^6\ldots.$$

Multiply up by e^x-e^{-x}, expand both sides, and write down the condition that the coefficient of x^{2p+1} is the same on both sides, and we get equation VI.

Generalisation of the "Orthopole" and Allied Theorems.

By Dr J. A. Third.

(*Read and Received 14th March 1913.*)

The greater part of this paper consists of generalisations of well-known theorems regarding perpendiculars to the sides of a triangle, or other base-lines, the perpendiculars being replaced by isoclinals.

To save confusion and secure generality, the following conventions will be observed:

(i) The angle which a line falling on a base-line makes with the latter will be taken as the angle, acute or obtuse as the case may be, obtained by measuring the shortest way round from the base-line to the first line (produced if necessary) in the *positive* or counter-clock sense.

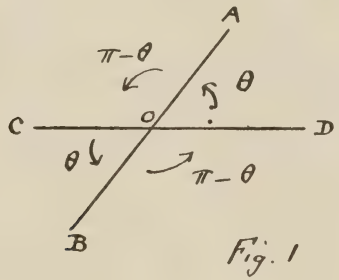

Fig. 1

Thus, in the figure, AO and BO make an angle θ with CD, but CO and DO make an angle $\pi - \theta$ with AB.

(ii) The areas of triangles will be regarded as positive when their vertices are taken counter-clockwise, as negative when their vertices are taken clockwise.

The following two propositions, which are so simple that their proofs may be omitted, supply the foundation for most of the subsequent theory.

(1). If from a point P a line PL be drawn to a base-line making

with it an angle θ, and Q and R be any two points on the base-line,

$$PQ^2 - PR^2 = LQ^2 - LR^2 + 4\triangle PQR\cot\theta,$$

or

$$LQ^2 - LR^2 = PQ^2 - PR^2 - 4\triangle PQR\cot\theta.$$

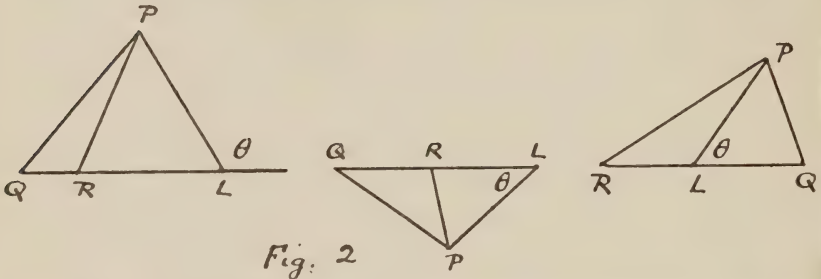

Fig. 2

This proposition as stated is universally true if (and not unless) we apply the two foregoing conventions and take the vertices of the triangle PQR in the following order—first, the vertex not lying on the base-line; and second, that vertex on the base-line which occurs first in the statement of the relation.

(2). If from any point O in the plane, lines OL, OM, ON be drawn to meet the sides BC, CA, AB respectively of a triangle ABC at the same angle θ,

$$LB^2 - LC^2 + MC^2 - MA^2 + NA^2 - NB^2 = -4\triangle ABC\cot\theta,$$

which may be written

$$[LMN] = -4\triangle ABC\cot\theta.$$

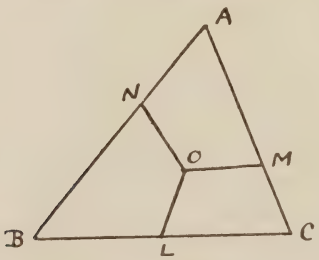

Fig. 3

This is an easy inference from (1), and is a generalisation of the standard theorem that if OL, OM, ON be perpendicular to the sides,

$$[LMN] = 0.$$

If L, M, N be *any* points on BC, CA, AB, a value of θ can always be found to satisfy the general relation, and by a *reductio ad absurdum* we can prove that the lines from these points making angles having this value with the sides, meet in a point. Thus we obtain a criterion of concurrency for isoclinals of which use will be made in what follows.

The theorem can be extended to the polygon, but the converse in this case is not necessarily true, and hence cannot be used as a criterion of concurrency.

"Isological" Triangles.

(3). It is well known that if the three perpendiculars from the vertices of one triangle to the sides of another are concurrent, the three corresponding perpendiculars from the vertices of the latter to the sides of the former are also concurrent. Two such triangles are said to be reciprocally *orthological.**

This theorem may be generalised as follows: If ABC, A′B′C′ be two triangles such that lines AL′, BM′, CN′ meeting B′C′, C′A′, A′B′ respectively at an angle θ, meet in a point O, then lines A′L, B′M, C′N meeting BC, CA, AB at the angle $\pi - \theta$, also meet in a point O′.

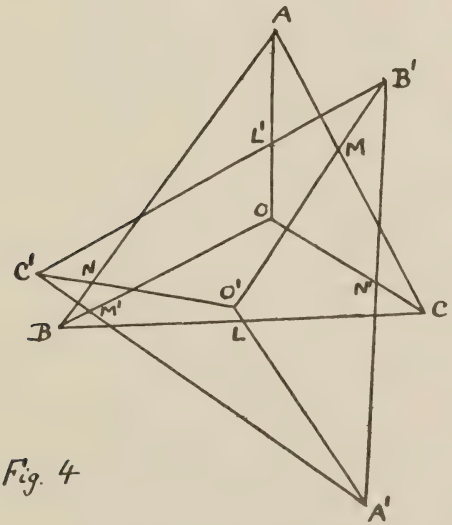

Fig. 4

* See a paper by Prof. Neuberg in *Mathesis*, 3rd series, vol. 1, p. 157.

Proof. Join A′ to B and C; B′ to C and A; C′ to A and B.

Then, using (1) and remembering the convention regarding the areas of triangles, we have

$$\begin{aligned}
[LMN] &= A'B^2 - A'C^2 - 4\triangle A'BC\cot(\pi-\theta)\\
&\quad + B'C^2 - B'A^2 - 4\triangle B'CA\cot(\pi-\theta)\\
&\quad + C'A^2 - C'B^2 - 4\triangle C'AB\cot(\pi-\theta)\\
&= AC'^2 - AB'^2 + BA'^2 - BC'^2 + CB'^2 - CA'^2\\
&\quad - 4(\triangle A'BC + \triangle B'CA + \triangle C'AB)\cot(\pi-\theta)\\
&= L'C'^2 - L'B'^2 + 4\triangle AC'B'\cot\theta\\
&\quad + M'A'^2 - M'C'^2 + 4\triangle BA'C'\cot\theta\\
&\quad + N'B'^2 - N'A'^2 + 4\triangle CB'A'\cot\theta\\
&\quad - 4(\triangle A'BC + \triangle B'CA + \triangle C'AB)\cot(\pi-\theta)\\
&= -(L'B'^2 - L'C'^2 + M'C'^2 - M'A'^2 + N'A'^2 - N'B'^2)\\
&\quad - 4(\triangle AC'B' + \triangle BA'C' + \triangle CB'A' + \triangle A'BC + \triangle B'CA\\
&\qquad + \triangle C'AB)\cot(\pi-\theta)\\
&=, \text{ by (2), } 4\triangle A'B'C'\cot\theta\\
&\quad - 4(\triangle AC'B' + \triangle BA'C' + \triangle CB'A' + \triangle A'BC + \triangle B'CA\\
&\qquad + \triangle C'AB)\cot(\pi-\theta)\\
&= -4(\triangle A'B'C' + \triangle AC'B' + \triangle BA'C' + \triangle CB'A'\\
&\qquad + \triangle A'BC + \triangle B'CA + \triangle C'AB)\cot(\pi-\theta)\\
&= -4\triangle ABC\cot(\pi-\theta).
\end{aligned}$$

Therefore, by (2), A′L, B′M, C′N meet in a point O′.

For convenience of reference a pair of triangles related as ABC and A′B′C′ are, will be said to be reciprocally *isological* with respect to the angle which the isoclinals from the vertices of the first-mentioned make with the sides of the second.

(4). If A′B′C′ be isological with ABC and BCA with respect to the same angle θ, it is also isological with CAB with respect to that angle.

θ being the angle of inclination throughout, let

A′L, B′M, C′N be isoclinals to BC, CA, AB respectively,

A′L′, B′M′, C′N′ „ „ „ CA, AB, BC „

and A′L″, B′M″, C′N″ „ „ „ AB, BC, CA „

If the first set meet in a point O, and the second in a point O′, we have to prove that the third set also meet in a point O″.

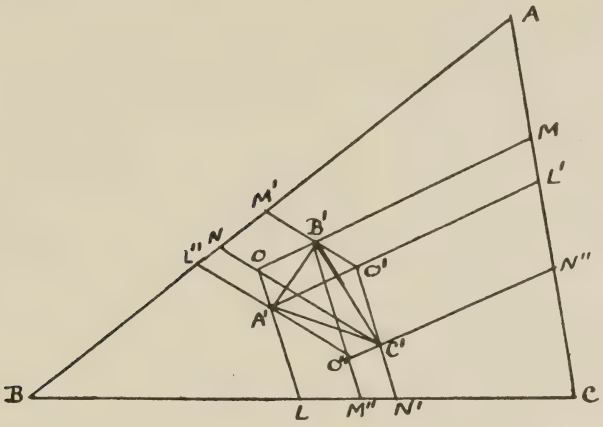

Fig. 5

We have, by (2) and (1),

$$-4\triangle ABC\cot\theta = [LMN]$$
$$= A'B^2 - A'C^2 - 4\triangle A'BC\cot\theta$$
$$+ B'C^2 - B'A^2 - 4\triangle B'CA\cot\theta$$
$$+ C'A^2 - C'B^2 - 4\triangle C'AB\cot\theta.$$

Also
$$-4\triangle ABC\cot\theta = [N'L'M']$$
$$= C'B^2 - C'C^2 - 4\triangle C'BC\cot\theta$$
$$+ A'C^2 - A'A^2 - 4\triangle A'CA\cot\theta$$
$$+ B'A^2 - B'B^2 - 4\triangle B'AB\cot\theta.$$

Adding and using (1) we have

$$-8\triangle ABC\cot\theta = B'C^2 - B'B^2 + C'A^2 - C'C^2 + A'B^2 - A'A^2$$
$$- 4(\triangle A'BC + \triangle A'CA + \triangle B'CA + \triangle B'AB$$
$$+ \triangle C'AB + \triangle C'BC)\cot\theta$$
$$= M''C^2 - M''B^2 + 4\triangle B'CB\cot\theta$$
$$+ N''A^2 - N''C^2 + 4\triangle C'AC\cot\theta$$
$$+ L''B^2 - L''A^2 + 4\triangle A'BA\cot\theta$$
$$- 4(3\triangle ABC + \triangle A\,BA + \triangle B'CB + \triangle C'AC)\cot\theta$$
$$= -[M''N''L''] - 12\triangle ABC\cot\theta.$$

Therefore $[M''N''L''] = -4\triangle ABC\cot\theta$.

Therefore, by (2), A'L'', B'M'', C'N'' meet in a point.

We can therefore assert that if two triangles be doubly isological, in the manner stated, with respect to a certain angle, they are triply isological with respect to that angle.

(5). Any two triangles ABC, A'B'C' are, in general, reciprocally isological in six ways.

Suppose that the vertices as named lie the same way round; we can then make the following statements:

(i) If through the extremities of BC, CA, AB three circles be drawn such that their segments on the *same* sides of these lines *as* A, B, C respectively, contain angles respectively equal to $\pi - A'$, $\pi - B'$, $\pi - C'$, the circles meet in a point O_1, such that AO_1, BO_1, CO_1 make with B'C', C'A', A'B' respectively the same angle θ_1, *i.e.* ABC is isological with A'B'C' with respect to θ_1.

(ii) The same statement as the foregoing, with cyclical interchange of A', B', C' and with O_2 for O_1 and θ_2 for θ_1.

(iii) The same as the foregoing, with further cyclical interchange of A', B', C' and with O_3 for O_2 and θ_3 for θ_2.

(iv) If through the extremities of BC, CA, AB three circles be drawn such that their segments on the *opposite* sides of these lines *from* A, B, C respectively, contain angles respectively equal to $\pi - A'$, $\pi - C'$, $\pi - B'$, the circles meet in a point O_4 such that AO_4, BO_4, CO_4 make with C'B', B'A', A'C' respectively the same angle θ_4, *i.e.* ABC is isological with A'C'B' with respect to θ_4. (Note that here the vertices of A'C'B' as named lie the *opposite* way round *from* those of ABC.)

(v) The same statement as the foregoing, with cyclical interchange in reverse order of A', B', C' and with O_5 for O_4 and θ_5 for θ_4.

(vi) The same as the foregoing, with further cyclical interchange, still in reverse order, of A', B', C' and with O_6 for O_5 and θ_6 for θ_5.

If the vertices of ABC and A'B'C' instead of lying the same way round lie in opposite ways round, the above statements must be modified by the interchange of the expressions *same as* and *opposite from* wherever they occur.

The truth of the statements, which need not be demonstrated in detail, will be manifest from an examination of the cyclic quadrilaterals in the figures of the different cases.

It will be noticed that O_1, O_2, O_3, O_4, O_5, O_6 are dependent only on the position of ABC and the shape of A′B′C′, and therefore remain invariable when ABC is fixed and A′B′C′ retains a constant shape, however much the latter triangle may vary in position or size. The respective conjugate points of concurrence of the lines from the vertices of A′B′C′ making angles $\pi - \theta_1$, $\pi - \theta_2$, etc., with the sides of ABC, are related to A′B′C′ in the same way as their conjugates are related to ABC, and consequently vary in position with A′B′C′.

If the triangles ABC, A′B′C′ are *directly* similar, O_1, O_2, O_3, O_4, O_5, O_6 become respectively the orthocentre, the negative Brocard point, the positive Brocard point, and the vertices A, B, C of ABC.

If the triangles ABC, A′B′C′ are *inversely* similar, O_1 is *any* point on the circumcircle of ABC, since the three circles giving rise to it coalesce in that circle. Hence we have the theorem which will be used later on, that any two inversely similar triangles are reciprocally isological with respect to every angle lying between 0 and π, *i.e.* in an infinite number of ways, and that, as the angle varies, the point of concurrence of the isoclinals from the vertices of either to the corresponding sides of the other has as locus the circumcircle of the former.

The positions assumed by the O-points in the other cases of inverse similarity corresponding to (ii), (iii), (iv), (v) and (vi) above, do not offer any special features of interest.

Orthopole and "Isopoles."

(6). It is well known that if perpendiculars AP, BQ, CR be drawn from the vertices of a triangle ABC to a base-line XY, and perpendiculars PL, QM, CN be drawn from P, Q, R to BC, CA, AB respectively, PL, QM, CN meet in a point O, which is called the *orthopole* of XY.*

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* See a paper on the Orthopole by Mr William Gallatly (Glasgow University Press).

This theorem may be generalised as follows: If isoclinals AP, BQ, CR be drawn from the vertices of a triangle ABC to a base-line XY making with it any angle θ, and if isoclinals PL, QM, CN be drawn from P, Q, R to BC, CA, AB respectively, making with them an angle $\pi - \theta$, PL, QM, CN meet in a point O, which may be called the *isopole* of XY with respect to the angle θ adopted.

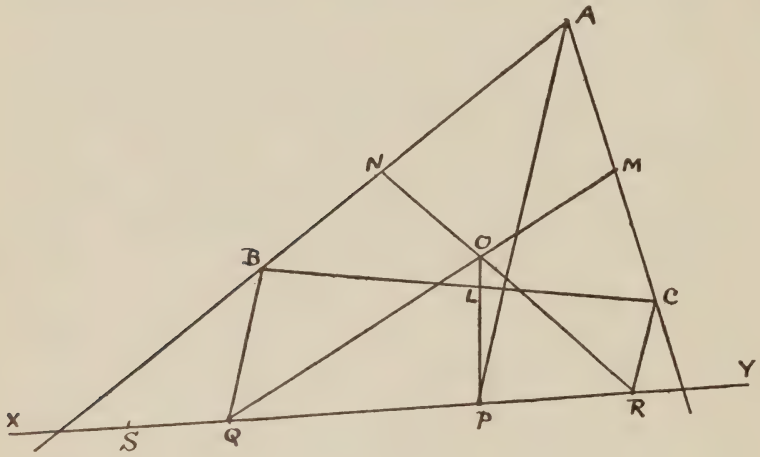

Fig. 6

Proof. Using (1), and remembering the convention regarding the areas of triangles, we have

$$\begin{aligned}
[\mathrm{LMN}] &= \mathrm{PB}^2 - \mathrm{PC}^2 - 4\triangle\mathrm{PBC}\cot(\pi-\theta) \\
&\quad + \mathrm{QC}^2 - \mathrm{QA}^2 - 4\triangle\mathrm{QCA}\cot(\pi-\theta) \\
&\quad + \mathrm{RA}^2 - \mathrm{RB}^2 - 4\triangle\mathrm{RAB}\cot(\pi-\theta) \\
&= \mathrm{AR}^2 - \mathrm{AQ}^2 + \mathrm{BP}^2 - \mathrm{BR}^2 + \mathrm{CQ}^2 - \mathrm{CP}^2 \\
&\quad - 4(\triangle\mathrm{PBC} + \triangle\mathrm{QCA} + \triangle\mathrm{RAB})\cot(\pi-\theta) \\
&= \mathrm{PR}^2 - \mathrm{PQ}^2 + 4\triangle\mathrm{ARQ}\cot\theta \\
&\quad + \mathrm{QP}^2 - \mathrm{QR}^2 + 4\triangle\mathrm{BPR}\cot\theta \\
&\quad + \mathrm{RQ}^2 - \mathrm{RP}^2 + 4\triangle\mathrm{CQP}\cot\theta \\
&\quad - 4(\triangle\mathrm{PBC} + \triangle\mathrm{QCA} + \triangle\mathrm{RAB})\cot(\pi-\theta) \\
&= -4(\triangle\mathrm{ARQ} + \triangle\mathrm{BPR} + \triangle\mathrm{CQP} + \triangle\mathrm{PBC} \\
&\qquad\qquad + \triangle\mathrm{QCA} + \triangle\mathrm{RAB})\cot(\pi-\theta) \\
&= -4\triangle\mathrm{ABC}\cot(\pi-\theta).
\end{aligned}$$

Therefore, by (2), PL, QM, RN are concurrent.

(7). The theory of isopoles could, if it were worth while, be considerably developed. I have had time only for a very cursory and superficial examination of the subject. The present section contains a selection of the results obtained, which are either loosely general in character or refer to the more obvious particular cases.

(i) If the base-line XY (Fig. 6) move parallel to itself, the locus of its isopole O with respect to a constant angle θ is a line making an angle θ with the fixed direction of XY.*

This follows from the obvious fact that, in the case supposed, the triangle QOR is always congruent to itself.

It follows also from the constant self congruence of QOR that the distance of O from XY is constant, and thus that the distance between a line and its isopole with respect to a given angle is unaltered by the movement of the line parallel to itself.

(ii) If XY (Fig. 6) vary in position while AP, BQ, CR remain fixed in direction thus making a variable angle θ with XY, and O be the isopole of XY in any position with respect to the value of θ for that position, the triangles QOR, ROP, POQ remain constant in shape; *e.g.* if ϕ be the constant angle made by AP with BC, then according as ϕ is less or greater than $\pi - C$, the angles at Q, R and O of the triangle QOR are respectively $\pi - C - \phi$, $\pi - B + \phi$ and $\pi - A$, or $-\pi + C + \phi$, $\pi + B - \phi$ and A. Thus if XY rotate round a fixed point S, since QR passes through this point and Q and R move on parallel straight lines, O will also move on a straight line, viz. the line joining the fixed second points of intersection of the circles SQO and SRO with BQ and CR respectively.

(iii) If we take O as the origin (Fig. 6) and $y = \mu x + \nu$ as the Cartesian equation of the line QR whose isopole with respect to a particular angle θ is O, and then form the equations of OQ and OR and hence those of BQ and CR, we find from the condition that BQ and CR make an angle θ with QR, taking β and γ as the angles made with the x-axis by AB and AC and (b_1, b_2), (c_1, c_2) as the coordinates of B and C, that

$$\nu\{\tan\theta - \tan\gamma - \theta + \mu(1 + \tan\theta \,.\, \tan\overline{\gamma - \theta})\}$$
$$= \mu^2(b_1 + b_2\tan\theta) + \mu\{b_1(\tan\theta - \tan\overline{\gamma - \theta}) - b_2(1 + \tan\theta \,.\, \tan\overline{\gamma - \theta})\}$$
$$+ (b_2 - b_1\tan\theta) \,.\, \tan\overline{\gamma - \theta},$$

with a similar equation having c for b and β for γ.

* The corresponding statement for the orthopole is given in Mr Gallatly's paper, p. 4.

Hence we obtain a cubic for the determination of μ, while for every value of μ there is only one value of ν.

Thus we conclude that, while every base-line has only one isopole for a particular value of θ, every point is in general the isopole, with respect to any given value of θ, of three different lines.

For example, the orthocentre H of the triangle ABC is obviously the orthopole of each of the three sides. Combining this statement with (i) above, we have the result that the locus of the orthopole of a line which moves parallel to one of the sides is the altitude corresponding to that side, and that the distance of the orthopole from the line is always equal to HD, where D is the foot of the altitude on the side.

One other example may be given. D is the orthopole of AD, of the line parallel to BC through the image of H with respect to BC, and, as might easily be shown, of the line joining A to the circumcentre.

(iv) If a line make an angle θ with a side of the triangle, it is obvious from a figure that the isopole of the line with respect to $\pi - \theta$ is the point where the line cuts that side.

(v) The locus of the isopoles, with respect to a given angle θ, of the lines passing through a fixed point S is in general a conic.

Take S as the origin and $y = x \,.\, \tan\phi$ as the equation of the variable base-line. Let also (b_1, b_2), (c_1, c_2) be the coordinates of B and C, and β, γ the angles made by AB and AC with the x-axis.

Then, since BQ and CR (Fig. 6) make an angle $\theta + \phi$, and QM and RN angles $\gamma - \theta$ and $\beta - \theta$ with the x-axis, we have as the equation of QM,

$$\tan^2\phi(x\tan\theta \,.\, \tan\overline{\gamma - \theta} - y\tan\theta + b_1 + b_2\tan\theta)$$
$$+ \tan\phi\{b_1(\tan\theta - \tan\overline{\gamma - \theta}) - b_2(1 + \tan\theta \,.\, \tan\overline{\gamma - \theta})\}$$
$$+ x\tan\theta \,.\, \tan\overline{\gamma - \theta} - y\tan\theta - (b_1\tan\theta - b_2)\tan\overline{\gamma - \theta} = 0,$$

with a similar equation for RN having β for γ and c for b.

Eliminating $\tan\phi$ between these equations, we have as the locus of O a conic in the form,

$$(ax + by + c)^2 - (dx + ey + f)(dx + ey + g) = 0.$$

It obviously touches the parallel lines (*def*) and (*deg*) at the points where these are met by (*abc*), and its centre lies on (*abc*) midway between the former two lines.

If from S lines SX, SY, SZ be drawn to the sides BC, CA, AB respectively of the triangle, making with them an angle $\pi - \theta$, it follows from (iv) that the conic passes through X, Y, Z. If S be the negative (or positive) Brocard point, and θ the Brocard angle ω (or $\pi - \omega$), X, Y, Z coincide with C, A, B respectively (or B, C, A), and the conic becomes a circumconic.

If S be the circumcentre and $\theta = \frac{\pi}{2}$, so that for *isopole* we have *orthopole*, the conic becomes the nine-point circle.* In this case the orthopoles of the perpendiculars from S to the sides are, by (iv), the mid-points of the sides; the orthopoles of the parallels through S to the sides are, by (iii), the mid-points of the lines joining the orthocentre to the vertices; and the orthopoles of the lines joining S to the vertices are the feet of the altitudes.

When S coincides with a vertex of the triangle, it is evident from a figure that the locus of the isopole of a variable line through S with respect to a given angle θ, degenerates into the line through the vertex making with the opposite side the angle $\pi - \theta$.

(vi) The locus of the isopole of a given line with respect to an angle θ which varies from zero to π is, in general, a cubic.

The equation of this cubic, if we take the given base-line as the axis of x, and (a_1, a_2), (b_1, b_2), (c_1, c_2) as the coordinates of A, B, C, is found to be

$$\begin{aligned}
&[x \,.\, \Sigma a_1 a_2(b_2 - c_2) - y \,.\, \Pi(b_2 - c_2) + \Sigma a_1 a_2(b_1 c_2 - b_2 c_1)]^2 \\
&- [x \,.\, \Pi(b_2 - c_2) + y \,.\, \Sigma a_1 a_2(b_2 - c_2) + \Sigma a_1 b_2 c_2(b_2 - c_2)] \\
&\times [(x^2 + y^2) \,.\, \Sigma a_1(b_2 - c_2) - x \,.\, \Sigma a_1^2(b_2 - c_2) \\
&\qquad + y\{\Sigma a_1(b_2^2 - c_2^2) - \Sigma a_1 a_2(b_2 - c_2) + \Pi(b_1 - c_1)\} \\
&\qquad - \Sigma a_2 b_1 c_1(b_1 - c_1)] = 0.
\end{aligned}$$

It is evident from (iv) that the cubic passes through the three points where the base-line cuts the sides of the triangle.

In the particular case when the base-line coincides with a side

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* Proved in Mr Gallatly's paper, p. 5.

of the triangle, the cubic degenerates into the circle passing through the extremities of that side and the orthocentre H.

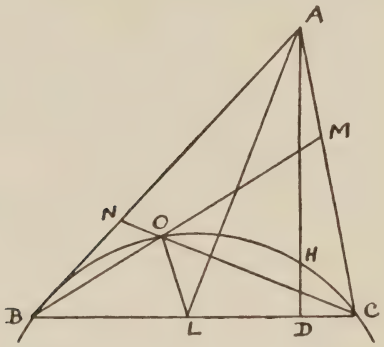


Fig 7

Let O, for example, be the isopole of BC with respect to any angle θ, and let BO, CO meet CA, AB in M and N respectively. Then BM, CN make an angle $\pi - \theta$ with CA, AB. Therefore AMON is a cyclic quadrilateral. Hence according as O lies on the same side of BC as A or on the opposite side, angle $BOC = \pi - A$ or A, so that O lies on the circle BHC. It is obvious that O is the image with respect to BC of the point where the circumcircle of ABC is met by a line from A making an angle θ with BC. Further, O and the two corresponding points for CA and AB are the three points into which the orthocentre breaks up when θ is substituted for $\frac{\pi}{2}$, and so may be regarded as constituting a kind of generalised orthocentre.

In connection with the case just mentioned it may be observed that if O_a, O_a' be the isopoles of BC with respect to θ and $\pi - \theta$, θ being here taken as acute as may without loss of generality be done, and O_b, O_b' and O_c, O_c' be the corresponding points for CA and AB, θ being the same throughout, then these six points not only lie in pairs on the circles BHC, CHA, AHB, but all lie on the circle with centre H and radius $2R\cos\theta$.

For the lines BO_a, BO_a' make angles $\pi - \theta$ and θ with CA. Hence angle $O_aBO_a' = \pi - 2\theta$, and is bisected by BH. Therefore,

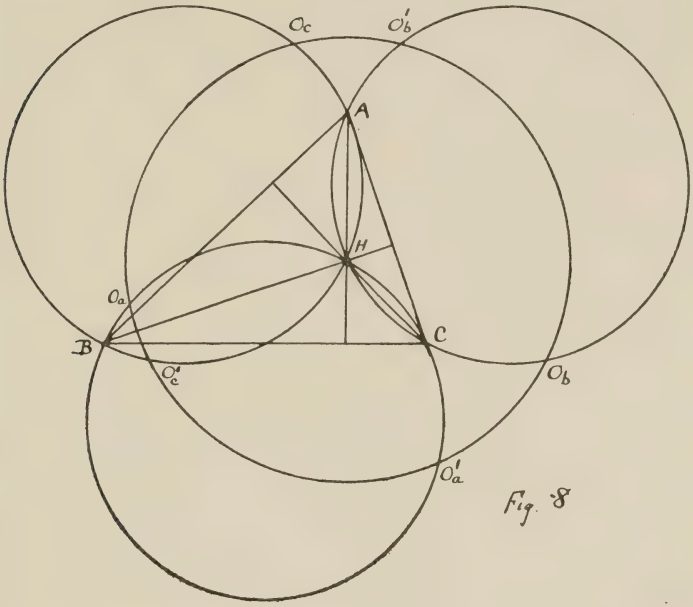


Fig. 8

since O_a, O_a' lie on the circle BHC,

$$O_aH = O_a'H = 2R\sin\left(\frac{\pi}{2} - \theta\right) = 2R\cos\theta.$$

Similarly the distance of H from any of the other O-points may be proved equal to $2R\cos\theta$.

Among the numerous other properties of this figure the following may be noticed $O_aO_a' = O_bO_b' = O_cO_c' = 2R\sin 2\theta$, 2θ being the angle which each subtends at H. The triangles $O_aO_bO_c$ and $O_a'O_b'O_c'$ are congruent, similar to ABC and circumscribed to it; the area of each $= 4\triangle ABC\cos^2\theta$. The lines O_bO_c, $O_b'O_c'$ meet the circumcircle of ABC in the images with respect to BC of O_a', O_a, with two similar statements. When $\theta = 0$, the pairs O_a, O_a'; O_b, O_b'; O_c, O_c' coincide in the points where the circle with centre H and radius 2R touches the circles BHC, CHA, AHB.

*Generalised Brocard Points and Circle.**

(8). I now return for a little to the figure of section (2) (Figure 3), where from a point O isoclinals OL, OM, ON are drawn to the sides BC, CA, AB respectively of the triangle, making with these an angle θ. If AL, BM, CN make angles α, β, γ with BC, CA, AB respectively, then

$$\cot\theta = \cot\alpha + \cot\beta + \cot\gamma.$$

For from (1) we have

$$LB^2 - LC^2 = AB^2 - AC^2 - 4\triangle ABC\cot\alpha,$$

with two similar statements for β and γ.

Hence $[LMN] = -4\triangle ABC(\cot\alpha + \cot\beta + \cot\gamma)$.

But by (2), $[LMN] = -4\triangle ABC\cot\theta$.

Therefore, $\cot\theta = \cot\alpha + \cot\beta + \cot\gamma$.

In all cases where $\theta = \frac{\pi}{2}$, as for example when LMN is a true Wallace line, or AL, BM, CN are the medians, the relation becomes

$$\cot\alpha + \cot\beta + \cot\gamma = 0.$$

When L, M, N coincide with B, C, A respectively (or C, A, B) O becomes the positive (or negative) Brocard point, θ becomes the Brocard angle ω (or $\pi - \omega$), and we obtain the well known relation,

$$\cot\omega = \cot A + \cot B + \cot C.$$

Thus, being given the position of L, M, N, we can regard O as the generalised Brocard point, and θ as the generalised Brocard angle, for that position. While it is obvious from (2) that for any given position of L, M, N there is only one generalised Brocard point O, and one generalised Brocard angle θ, it is clear that for any given position of O there are an infinite number of possible values of θ, and an infinite number of different positions of L, M, N, viz. the different positions assumed by the vertices of an inscribed triangle of constant species of which O is the centre of similitude. Hence any point whatever in the plane can be regarded

* There have been various generalisations of the Brocard circle, *e.g.* that of Mr Griffiths in the *Proceedings of the London Mathematical Society*, Vol. XXV., pp. 75–85 and 376–388, and Vol. XXVI., pp. 173–183: but all those known to me differ widely from the one now given.

as a generalised Brocard point, and with it can be associated as generalised Brocard angle any angle whatever lying between zero and π, thus determining the position of L, M, N.

The position of L, M, N being given, we can easily deduce from the relation given above connecting θ, α, β, γ, a geometrical construction determining the position of O.

When L, M, M are collinear the locus of O, as is well known, is the circumcircle of ABC, LMN being then the true or generalised (isoclinals replacing perpendiculars) Wallace line of O. When the triangle LMN is of constant shape, O, as is also well known, is fixed. It might readily be shown by using trilinear coordinates that when AL, BM, CN are concurrent, the locus of O for any given value of θ is a cubic. The coordinates of O being given, with the condition that AL, BM, CN shall be concurrent, we find a cubic equation for the determination of $\cot\theta$; hence for every position of O there are three positions of L, M, N such that AL, BM, CN are concurrent; of these positions one is always real, and for certain positions of O, but not for all, the other two also. When of two different positions of L, M, N the one set are the isotomic conjugates of the other, it is evident from (2) that the corresponding generalised Brocard angles are supplementary; this is of course the case with the true Brocard points.

The connection of all this with the well known "Point O" theorem is obvious, O being the point of intersection of the circles AMN, BNL, CLM.

It may not be amiss to give the coordinates of O for certain data. If $BL = l$, $LC = l'$, $CM = m$, $MA = m'$, $AN = n$, $NB = n'$, then the trilinear coordinates of O are

$$blm + cn'l' - all',\ cmn + al'm' - bmm',\ anl + bm'n' - cnn'.$$

If AL, BM, CN are concurrent in the point whose trilinear coordinates are λ, μ, ν, then the trilinear coordinates of O are

$$\frac{a}{b\mu + c\nu}\left(\frac{b\nu\lambda}{c\nu + a\lambda} + \frac{c\lambda\mu}{a\lambda + b\mu} - \frac{a\mu\nu}{b\mu + c\nu}\right),$$

with two similar expressions obtained by the cyclic interchange of a, b, c and λ, μ, ν.

(9). If O, O′ be any two generalised Brocard points, θ, θ' the corresponding generalised Brocard angles, and L, M, N and L′, M′, N′ the corresponding positions of L, M, N, and if LO, L′O′ meet in A′, MO, M′O′ in B′, and NO, N′O′ in C′, then (i) A′, B′, C′, O, O′ lie on a circle, the generalised Brocard circle; (ii) perpendiculars from A′, B′, C′ to the corresponding sides of ABC cointersect in a point S on the circle, parallels through A′, B′, C′ to the corresponding sides of ABC cointersect in a point K on the circle, and generally any isoclinals from A′, B′, C′ to the corresponding sides of ABC cointersect in a point on the circle; (iii) the triangle A′B′C′ (corresponding to Brocard's first triangle) is inversely similar to ABC; (iv) perpendiculars from A, B, C to the corresponding sides of A′B′C′ cointersect in a point on the circumcircle of ABC (corresponding to Tarry's point), and generally isoclinals from A, B, C to the corresponding sides of A′B′C′ cointersect in a point on the circumcircle of ABC; (v) A′B′C′ is in perspective with the triangles LMN, L′M′N′ with O, O′ as centres of perspective; (vi) $\triangle A'BC + \triangle B'CA + \triangle C'AB = \triangle ABC$, etc., etc. In fact the correspondence between the generalised and the true Brocard circles is complete.

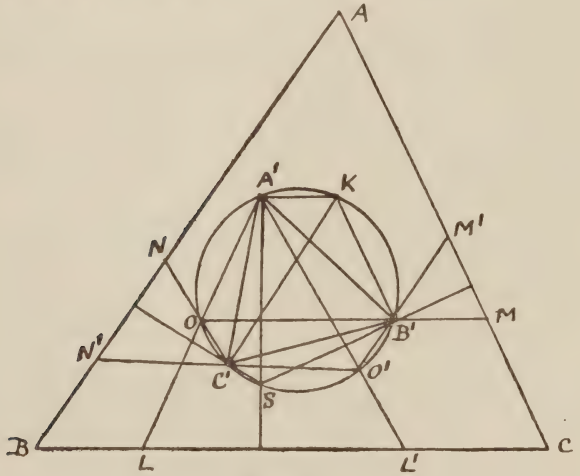


Fig. 9

Proof. $L\widehat{A'}L' = M\widehat{B'}M' = N\widehat{C'}N' = \theta \sim \theta'$. Hence the angles OA′O′, OB′O′, OC′O′ are equal or supplementary, which establishes (i).

Again, since AMON and A′B′OC′ are cyclic quadrilaterals, $C'\widehat{A}'B' = A$. Similarly it may be shown that $A'\widehat{B}'C' = B$ and $B'\widehat{C}'A' = C$. Thus (iii) is established.

(ii) and (iv) follow immediately from the fact referred to in (5) that if two triangles ABC, A′B′C′ be inversely similar they are reciprocally isological in an infinite number of ways, isoclinals from the vertices of either to the corresponding sides of the other co-intersecting in a point on the circumcircle of the former.

(v) is at once obvious from the figure.

(vi) is evident from the fact that the triangles A′BC, B′CA, C′AB are respectively equal to the triangles KBC, KCA, KAB.

When $\theta = \theta'$, the generalised Brocard circle obviously degenerates into the line OO′ and the line at infinity.

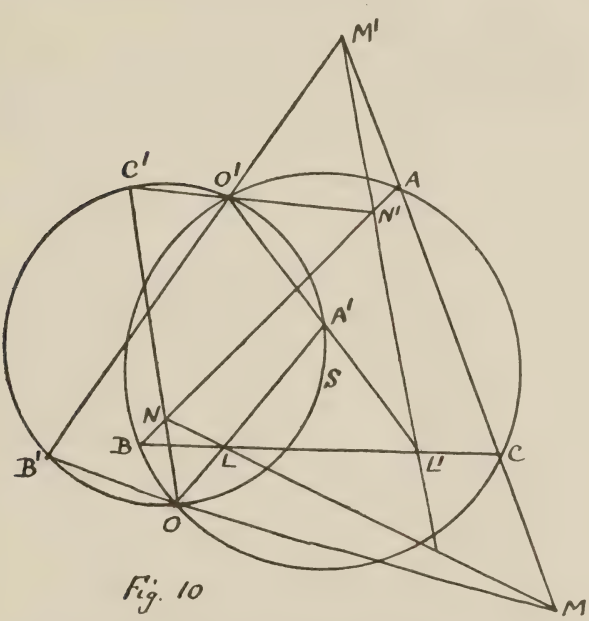


Fig. 10

When the pairs L, L′ ; M, M′ ; N, N′ are isotomic conjugates, and consequently θ and θ' are supplementary, S is obviously the circumcentre of ABC, as is the case for the true Brocard circle.

When in addition to the isotomic relation, L, M, N are collinear, and therefore also L′, M′, N′, then not only do O, O′ lie on the circumcircle of ABC (LMN, L′M′N′ being their true or generalised Wallace lines), but OO′ subtends at the circumference of that circle an angle equal to θ, and the angle between LMN and L′M′N′ is also θ.

For in this case twice the angle subtended by OO′ at the circumference of the circumcircle $= \widehat{OSO'} = \widehat{OA'O'} = 2\theta$. Also, in the figure given, the angle between LMN and L′M′N′

$$= \widehat{BNL} - \widehat{BN'L'} = \theta + \widehat{ONL} - \pi + \widehat{BN'M'}$$
$$= \theta + \widehat{OBC} - \pi + \theta + \widehat{O'N'M'}$$
$$= 2\theta - \pi + \widehat{OBC} + \widehat{O'AM'}$$
$$= 2\theta - \pi + \widehat{OBC} + \widehat{O'BC}$$
$$= 2\theta - \pi + \widehat{OBO'} = 2\theta - \pi + \pi - \theta = \theta.$$

Any circle whatever in the plane can be taken as a generalised Brocard circle, and any two points O, O′ on it can be taken as the associated generalised Brocard points. We may fix A′ arbitrarily anywhere on this circle. Then by joining A′ to O, O′ we get L, L′ and so determine θ, θ'. The other points M, M′, N, N′, B′, C′ are then obtained by drawing isoclinals from O, O′ making angles θ, θ' respectively with CA and AB.

By treating them as generalised Brocard circles we can obtain many of the properties of the circles connected with the triangle. For example, if we regard the circumcircle as a generalised Brocard circle, we at once obtain the theorem (otherwise sufficiently obvious) that if LMN be the true or a generalised Wallace line of O, then OL, OM, ON meet the circle in the vertices of a triangle inversely congruent to ABC, and that while LMN moves parallel to itself the vertices of this triangle remain fixed.

A method of finding (i) the double points of a unicursal curve, (ii) unicursal quartics with three given double points.

By Dr R. J. T. Bell.

(*Read and Received 13th December 1912.*)

The equations

$$x=\frac{f_1(t)}{\phi(t)},\ y=\frac{f_2(t)}{\phi(t)},$$

where $f_1(t)$, $f_2(t)$, $\phi(t)$ are polynomials in t, determine a unicursal curve. If α and β can be found so that

$$x-\alpha=\frac{(at^2+2bt+c)\psi_1(t)}{\phi(t)} \text{ and } y-\beta=\frac{(at^2+2bt+c)\psi_2(t)}{\phi(t)},$$

then the point (α, β) is on the curve and is given by two values of t, t_1 and t_2, which are the roots of the equation $at^2+2bt+c=0$. The gradients of the tangents through (α, β) are

$$\lim_{t\to t_1}\frac{y-\beta}{x-\alpha} \text{ and } \lim_{t\to t_2}\frac{y-\beta}{x-\alpha},$$

or

$$\lim_{t\to t_1}\frac{\psi_2(t)}{\psi_1(t)} \text{ and } \lim_{t\to t_2}\frac{\psi_2(t)}{\psi_1(t)}.$$

Hence if t_1 and t_2 are real and distinct, there are two distinct tangents through (α, β), and (α, β) is a node. If $t_1=t_2$, the tangents coincide and (α, β) is a cusp. If t_1 and t_2 are imaginary, the tangents are imaginary and (α, β) is a conjugate point.

We have assumed that $\psi_1(t)$ and $\psi_2(t)$ have no common factor. If they had, (α, β) would be a multiple point of higher order than the second and the tangents through it could be found as above.

Since, when (x, y) is a double point,

$$f_1(t)-x\phi(t) \quad \text{and} \quad f_2(t)-y\phi(t)$$

have a common factor of the form $at^2+2bt+c$, we may find the double points as follows:—regard $f_1(t)-x\phi(t)$ and $f_2(t)-y\phi(t)$ as polynomials in t, and proceed to find their H.C.F. At a certain stage of the process the remainders will be of the form

$$ut^2+vt+w, \quad u't^2+v't+w',$$

where u, v, w, u', v', w' are functions of x and y. These remainders must both be multiples of the common factor $at^2+2bt+c$, and therefore

$$\frac{u}{u'}=\frac{v}{v'}=\frac{w}{w'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Solve these equations for x and y. Any values of x and y which satisfy all three equations give double points. (Of course, we may take instead of $f_1(t)-x\phi(t)$ and $f_2(t)-y\phi(t)$ any legitimate combination of these expressions which will simplify the process of finding the H.C.F.).

Let us consider as an example the curve

$$\frac{x}{a}=\frac{t^2}{t^3+t^2-1}, \quad \frac{y}{a}=\frac{-t^4-t^3}{t^3+t^2-1}.$$

Equating the values of t^3+t^2-1, we have

$$\left.\begin{aligned} xt^2+xt+y&=0.\\ \text{We have also}\qquad xt^3+(x-a)t-x&=0.\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Apply the rule for finding the H.C.F. to the expressions in equations (2).

$$\begin{array}{l|l|l} xt^2+xt+y & xt^3+(x-a)t-x & t \\ & xt^3+xt^2+yt & \\ \hline & -at^2-yt-x & \end{array}$$

Hence for double points

$$\frac{x}{a}=\frac{x}{y}=\frac{y}{x},$$

or

$$x(y-a)=0, \quad x^2=y^2, \quad x^2=ay.$$

The three equations are satisfied at $(0, 0)$, (a, a), $(-a, a)$, and these points are therefore double points. Since x and y have a common factor t^2, $(0, 0)$ is a cusp. The gradient of the tangent there is

$$\lim_{t\to 0}\frac{y}{x}=\lim_{t\to 0}(-t^2-t)=0,$$

hence OX is the tangent at the origin.

Again,
$$x-a=\frac{-a(t-1)(t^2+t+1)}{t^3+t^2-1},$$

$$x+a=\frac{a(t+1)(t^2+t-1)}{t^3+t^2-1},$$

$$y-a=\frac{-a(t^2+t-1)(t^2+t+1)}{t^3+t^2-1}.$$

Hence (a, a) is a conjugate point, being given by $t^2+t+1=0$, and $(-a, a)$ is a node, being given by $t^2+t-1=0$. The gradient of a tangent at the node is

$$\mathrm{L}\frac{y-a}{x+a} = \mathrm{L}\left(-\frac{t^2+t+1}{t+1}\right) \text{ when } t^2+t-1=0,$$

$$= \mathrm{L}\frac{-2}{t+1} = \mathrm{L}(-2t) = 3{\cdot}2 \text{ or } -1{\cdot}2.$$

The form of the curve is shown in Fig. 1. The numbers affixed are the values of t, and it is interesting to note in this and the

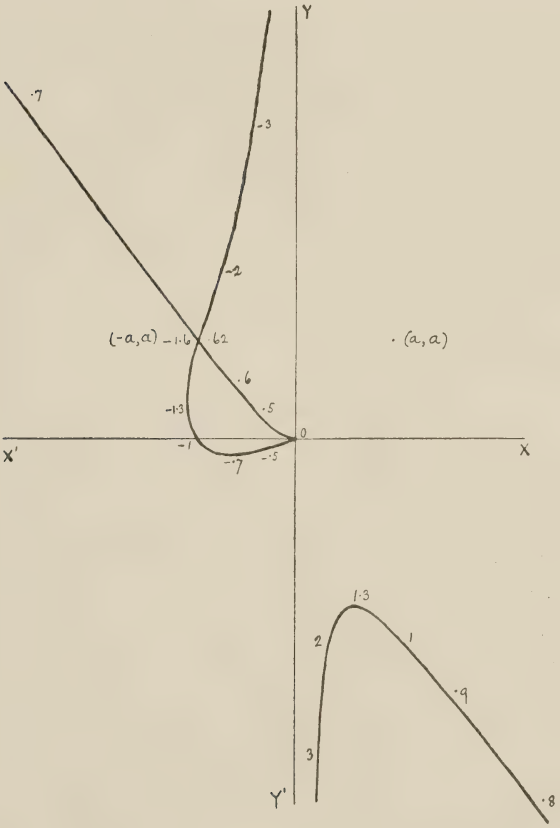


Fig. 1.

following examples how rapidly t varies at some parts of the curve and how slowly at others.

The constraint equation of the curve is found by eliminating t between the freedom equations, or between the equations (2). But since we may combine the expressions in these equations as we did in finding the H.C.F., the above elimination reduces to the elimination of t between the equations

$$xt^2+xt+y=0,\ at^2+yt+x=0,\ \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

formed by equating to zero the remainders of the second degree in the H.C.F. process. From (3)

$$\frac{t^2}{x^2-y^2}=\frac{t}{x^2-ay}=\frac{1}{x(y-a)},$$

whence the constraint equation is

$$(x^2-ay)^2=x(y-a)(x^2-y^2).$$

It is easy to verify generally that any solutions of *all* the three equations (1) give double points on the curve. As above, the constraint equation of the curve is found by eliminating t from the equations

$$ut^2+vt+w=0,\ u't^2+v't+w'=0,$$

and is therefore

$$(wu'-w'u)^2=(uv'-u'v)(vw'-v'w).$$

Hence any values of x and y which satisfy

$$wu'-w'u=0,\ uv'-u'v=0,\text{ and } vw'-v'w=0$$

give double points. It should be noted that though the three equations (1) are not independent it is possible to find values of x and y which satisfy two of them and do not satisfy the third. For example, any solutions of $u=0$ and $u'=0$ would satisfy the first two and might not satisfy the third. The constraint equation shows that if none of the expressions $wu'-w'u$, $uv'-u'v$, $vw'-v'w$ is a perfect square, (α, β) is a double point only if $x=\alpha$, $y=\beta$ satisfy all three equations (1).

The nature of the double point can be decided by an examination of the discriminants of the factors ut^2+vt+w, $u't^2+v't+w'$. If v^2-4wu is positive when the coordinates of the double point are

substituted in u, v, w, the double point is a node; if it is negative, a conjugate point. But if $v^2 - 4wu = 0$ we cannot at once infer that the double point is a cusp. For the coordinates (α, β) of the double point may satisfy the equations $u = v = w = 0$, and in this case the factor $ut^2 + vt + w$ disappears on substituting the coordinates. Hence

$$ut^2 + vt + w \text{ is of the form } \mathrm{F} \,.\, (pt^2 + qt + r)$$

where F is a factor which is zero when $x = \alpha$ and $y = \beta$, and the double point is a node, cusp, or conjugate point according as the roots of $pt^2 + qt + r = 0$ are real, equal, or imaginary. (If *both* of the discriminants $v^2 - 4wu$, $v'^2 - 4w'u'$ vanish when $x = \alpha$, $y = \beta$, (α, β) is generally a cusp.)

An example may help to make this clearer. Take the curve

$$x = a(t^3 - 2t),\ y = a(t^4 - 3t^2 + 1).$$

We have for the H.C.F.

$$\left.\begin{array}{l|l|l|l} t & \begin{array}{l} at^3 - 2at - x \\ at^3 - xt^2 + t(y - a) \end{array} & \begin{array}{l} at^4 - 3at^2 + (a - y) \\ at^4 - 2at^2 - xt \end{array} & t \\ \hline & t^2x - (y + a)t - x & -at^2 + tx + (a - y) & \end{array}\right.$$

therefore for double points $\dfrac{x}{a} = \dfrac{a + y}{x} = \dfrac{x}{a - y}$,

or $$x^2 + y^2 = a^2,\ xy = 0,\quad x^2 = a(y + a).$$

Whence we find $(a, 0)$, $(-a, 0)$, $(0, -a)$ are double points; [$(0, a)$ is not a double point, though its coordinates satisfy two of the equations].

The discriminants of the remainders are

$$(a + y)^2 + 4x^2,\ x^2 + 4a(a - y).$$

The first of these vanishes when $x = 0$, $y = -a$, but the second is positive. Since $x^2 = a(y + a)$ we may write the first remainder in the form

$$t^2x - t\frac{x^2}{a} - x$$

and we see that it vanishes on account of the factor x. Divide out by x/a and we get $at^2 - xt - a$, whose discriminant, $x^2 + 4a^2$ is always

positive, and hence $(0, -a)$ is a node, and the other double points are also nodes. Fig. 2 shows the form of the curve.

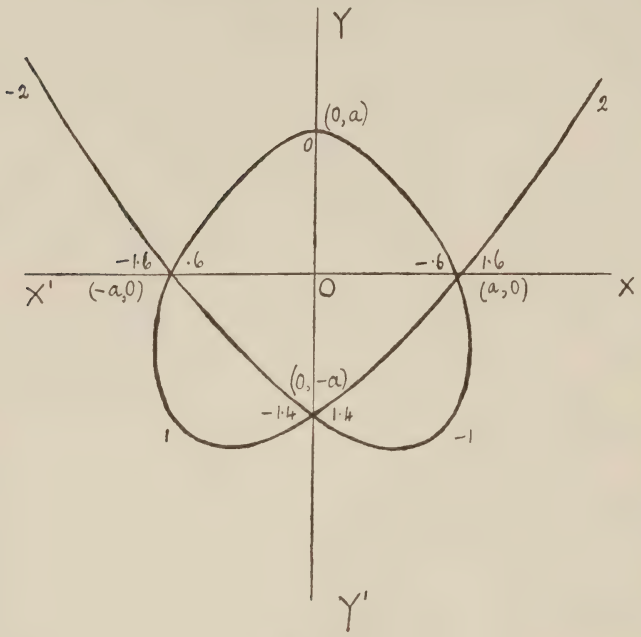


Fig. 2.

The foregoing discussion suggested a means of obtaining unicursal quartic curves with three given points as double points. If we can find two conics through the three points and their equations can be put in the forms $\frac{u}{u'} = \frac{v}{v'} = \frac{w}{w'}$, where u, v, w, u', v', w', are linear functions of x and y, then the coordinates of the three given points will generally satisfy all the three equations

$$\frac{u}{u'} = \frac{v}{v'} = \frac{w}{w'}.$$

If therefore we eliminate t from the equations

$$ut^2 + vt + w = 0,$$
$$u't^2 + v't + w' = 0,$$

we shall have the constraint equation of a quartic with the given

points as double points. If we arrange these equations as equations in x and y and solve, we express x and y as functions of t, and obtain the freedom equations of the curve.

It is easy to find the two conics. We may determine the constants in the equations

$$(x+a)(y+b)=c \quad \alpha y=x^2+\beta x+\gamma,$$

so that the conics they represent may pass through three given points. The second equation may then be written

$$\alpha(y-\delta)=(x+a)(x+\beta-a),$$

where $$\alpha\delta=\gamma-a\beta+a^2,$$

and the two equations therefore give

$$\frac{x+a}{c}=\frac{1}{y+b}=\frac{\alpha(y-\delta)}{c(x+\beta-a)}.$$

Thus if we take the points $(a, 0)$, $(-a, 0)$, $(0, -a)$ to be the double points, we may choose $x^2+y^2=a^2$, $x^2=a(y+a)$ as the conics. These may be written $\frac{x}{a+y}=\frac{a-y}{x}=\frac{a}{x}$, which gives the third equation $xy=0$. This is satisfied by the coordinates of the given points.

Consider now

$$xt^2+2kt(a-y)+la=0, \quad \ldots\ldots\ldots\ldots\text{(i)}$$
$$(a+y)t^2+2ktx+lx=0, \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

where k and l are arbitrary constants.

Eliminate t from these equations and we have

$$l(x^2-ay-a^2)+4k^2xy(x^2+y^2-a^2)=0.$$

Solve for x and y and we get

$$\frac{x}{a}=-\frac{4kt^3+lt^2}{t^4+4kt^2+2klt}, \quad \frac{y}{a}=\frac{(l+2kt)^2-t^4}{t^4+4kt^2+2klt}.$$

We have here the constraint and the freedom equations of a unicursal quartic with the three given points as double points.

The discriminant of (i) is $k^2(a-y)^2-lax\equiv D_1$, say,

,, ,, ,, (ii) ,, $k^2x^2-lx(a+y)\equiv D_2$,,

When $x=a$, $y=0$, $D_1=(k^2-l)a^2$, $D_2=(k^2-l)a^2$;

,, $x=-a$, $y=0$, $D_1=(k^2+l)a^2$, $D_2=(k^2+l)a^2$;

,, $x=0$, $y=-a$, $D_1=4k^2a^2$, $D_2=0$.

Hence $(0, -a)$ must be a node, but by choosing $l=k^2$ we can make $(a, 0)$ a cusp, or by $l>k^2$, a conjugate point.

By rearranging the coefficients of t^2, t, and the constant terms in (i) and (ii) in the cyclic order we obtain three curves.

Greater control over the nature of the double points may be obtained by the introduction of more arbitrary constants. Thus the equations of any conics through the three points $(a, 0)$, $(-a, 0)$, $(0, -a)$ may be written

$$x^2 + lxy = (a+y)(a+my),\ x^2 + \lambda xy = (a+y)(a+\mu y).$$

Whence
$$\frac{x}{a+y} = \frac{a+my}{x+ly} = \frac{a+\mu y}{x+\lambda y}.$$

The third conic is therefore

$$(a+my)(x+\lambda y) = (x+ly)(a+\mu y),$$

or
$$y[x(m-\mu) + y(m\lambda - l\mu) + a(\lambda - l)] = 0.$$

It passes through all three points if

$$l - \lambda + m\lambda - l\mu = 0.$$

Provided that l, m, λ, μ, are chosen to satisfy this equation, they are otherwise at our disposal and we may assign values to them so that the discriminants have given signs, or that the curves may satisfy other conditions.

The following examples have been constructed by the above methods:—

(i)
$$\frac{x}{a} = \frac{3t - 2t^3}{t^4 - 2t^2 + 2},\ \frac{y}{a} = \frac{t^4 - 3t^2 + 1}{t^4 - 2t^2 + 2};$$

$$x^2y^2 = (x^2 + y^2 - a^2)(x^2 + 2y^2 + ay - a^2);\quad \text{(Fig. 3).}$$

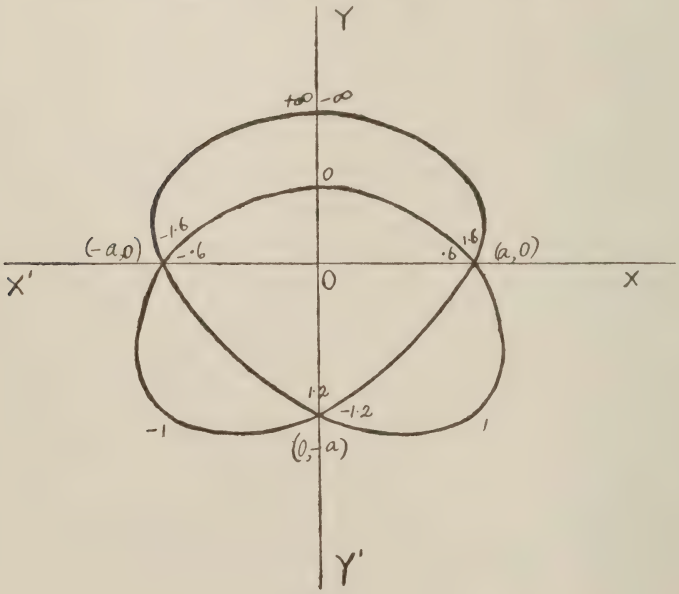


Fig. 3.

(ii) $$\frac{x}{a} = \frac{4t}{t^4+3}, \quad \frac{y}{a} = \frac{-2(t^2+1)}{t^4+3};$$

$$x^2(y+a)^2 + (x^2 - 3y^2 - 2ay)(x^2 - y^2) = 0; \qquad \text{(Fig. 4).}$$

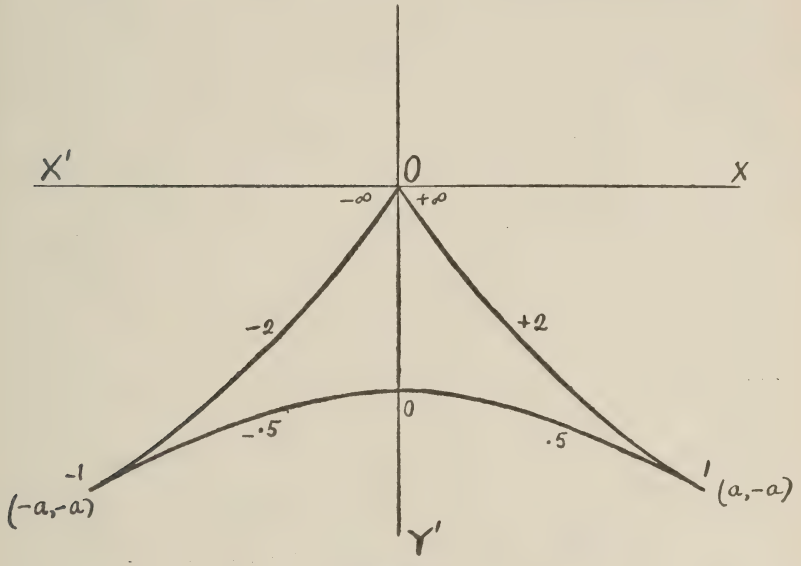


Fig. 4.

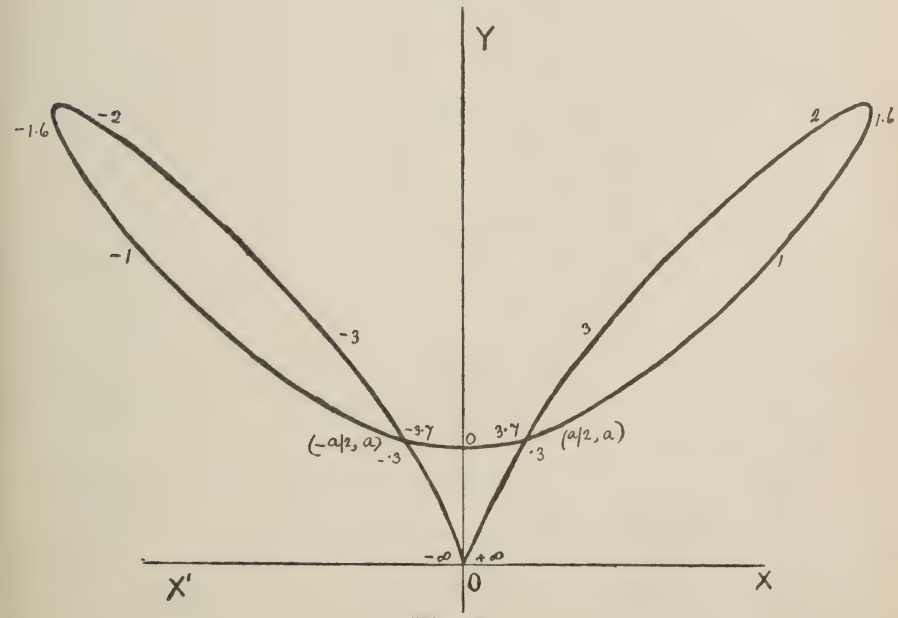


Fig. 5.

(iii) $$\frac{x}{a}=\frac{22t}{t^4-3t^2+12},\quad \frac{y}{a}=\frac{11(t^2+1)}{t^4-3t^2+12};$$

$$121x^2(y-a)^2+4(4x^2-12y^2+11ay)(4x^2-y^2)=0;\quad \text{(Fig. 5).}$$

(iv) $$\frac{x}{a}=\frac{2t(1-2t^2)}{4t^4-3t^2+1},\quad \frac{y}{a}=\frac{-(1+t^2)(1-2t^2)}{4t^4-3t^2+1};$$

$$9x^2(y-a)^2+8(2x^2-y^2-ay)(x^2-2y^2+ay)=0;\quad \text{(Fig. 6).}$$

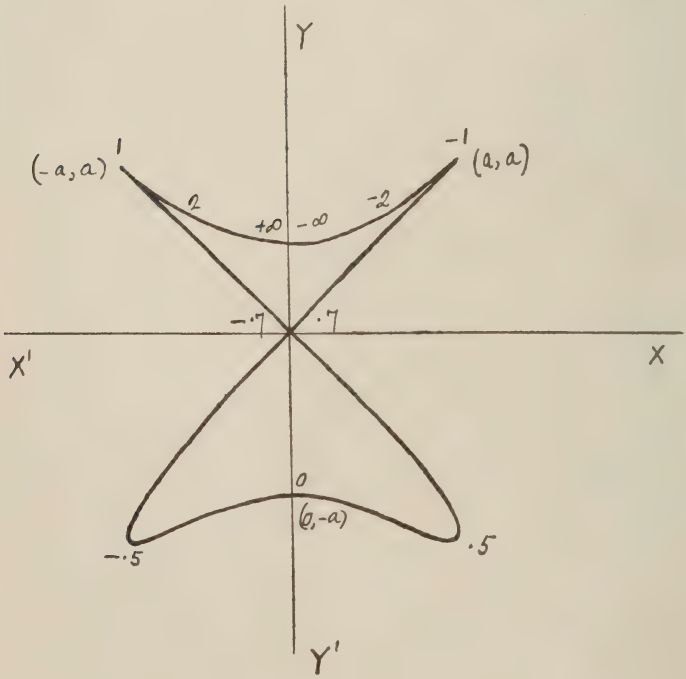


Fig. 6.

(v) $$\frac{x}{a}=\frac{t-2t^2}{t^4-t^3+1},\quad \frac{y}{a}=\frac{t^4-2t^3+t^2-1}{t^4-t^3+1};$$

$$(x^2+y^2-a^2)^2=xy(x^2-ay-a^2);$$

($(a, 0)$ is a conjugate point on this curve); (Fig. 7).

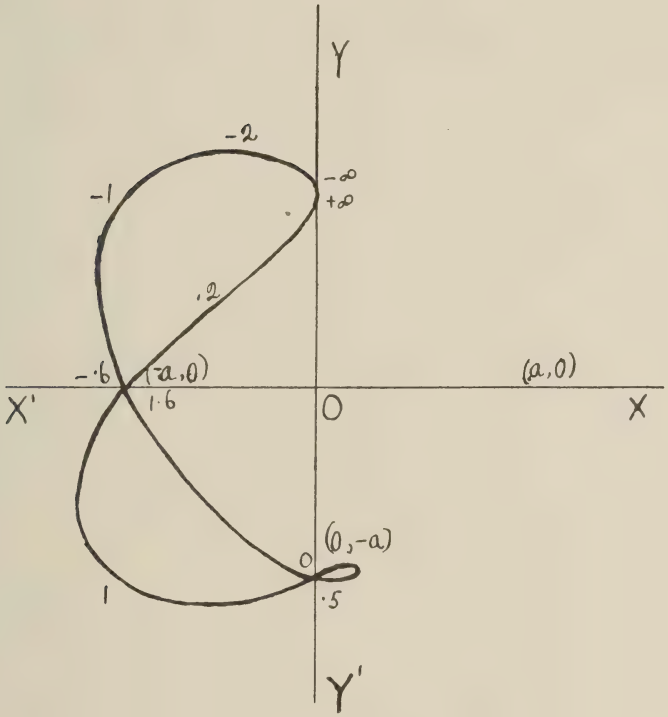


Fig. 7.

(vi) $$\frac{x}{a}=\frac{-t(3t-2)}{2t^4-4t^3+1},\quad \frac{y}{a}=\frac{t^2(t-2)^2-1}{2t^4-4t^3+1};$$

$$(x^2+2y^2+ay-a^2)^2=8xy(x^2-ay-a^2);\quad \text{(Fig. 8).}$$

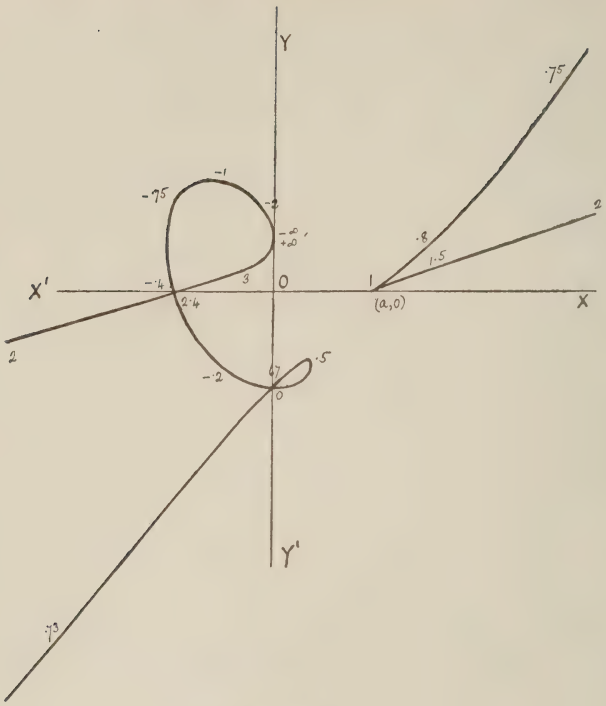


Fig. 8.

Notes on Mathematical Induction.

By Dr R. F. Muirhead.

(*Read 13th March 1913. Received 22nd April 1913.*)

Mathematical induction in its simplest form may be stated thus: Suppose there is a set of propositions $p_0, p_1, p_2 \ldots$ which are so related that the truth of p_n implies the truth of p_{n+1}, then if p_0 is true, it follows that all the other propositions of the set are true. For since p_0 is true therefore p_1 is true, therefore p_2 is true, and so on as far as we please.

A well-known example of the application of this method is the proof of the binomial theorem for a positive integral index. It is based on the identity

$$\left(1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots\right)(1+x) = 1 + (n+1)x + \frac{(n+1)n}{1.2}x^2 + \ldots . (1)$$

From this we deduce

$$\left\{1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots - (1+x)^n\right\}(1+x)$$

$$= 1 + (n+1)x + \frac{(n+1)n}{1.2}x^2 + \ldots - (1+x)^{n+1} \ldots\ldots (2)$$

Denoting by p_n the proposition that the first factor of the left side of this equation is identically zero, the proposition that the right side is identically zero will be denoted by p_{n+1}, and (2) shows that if p_n is true, so is p_{n+1}. The proposition p_0 asserts that $1 - (1+x)^0 = 0$, and is true. Hence p_n is true.

It will be convenient in what follows to denote propositions by equations of the form $p = 0$, in which p may be called the *error* of the proposition, so that $p = 0$ asserts that the error of the proposition is zero, or, in other words, that the proposition is true. In using this notation we do not imply that p is an actual algebraical expression (though very many propositions can be made to take the form of an algebraic expression equated to numerical zero). For

example, the letter p might stand for the error of the statement "N is a prime," so that $p=0$ would indicate the assertion "N is a prime."

Returning to our topic, we note that a slightly more complex form of mathematical induction is that which rests on a relation such as this: $p_n=0$ and $p_{n+1}=0$ together imply $p_{n+2}=0$, which combined with the assertions $p_0=0$ and $p_1=0$ enables us to assert that $p_2=0$, $p_3=0$, $p_4=0$ and so on, as far as we please.

As an example, let us prove that for all non-fractional values of x the expression $x^4-4x^3+5x^2-2x\equiv f(x)$ is divisible exactly by 12

$$f(x)=x(x-1)(x-2)(x-3)+2x(x-1)(x-2)$$

$$\therefore f(x+1)-f(x)=4x(x-1)(x-2)-6x(x-1)$$

$$\therefore f(x+2)-2f(x+1)+f(x)=12x(x-1)+12x=12x^2 \ldots\ldots\ldots(3).$$

Denoting by $p_x=0$ the proposition that $f(x)$ is divisible by 12, we see at once that (3) shows that $p_x=0$ and $p_{x+1}=0$ together imply $p_{x+2}=0$. But $f(0)=0$ and $f(1)=0$, so that $p_0=0$ and $p_1=0$. It follows that $p_2=0$, $p_3=0$, $p_4=0, \ldots p_x=0$, where x is any positive integer.

We remark that (3) also shows that $p_{x+1}=0$ and $p_{x+2}=0$ together imply $p_x=0$, hence we deduce also $p_{-1}=0$, $p_{-2}=0$, $\ldots p_x=0$ where x is any negative integer.

The next higher type of mathematical induction is when there exists a relation, true for all values of n, or for all integral values, between p_n, p_{n+1}, p_{n+2}, p_{n+3} in virtue of which we can infer that $p_{n+3}=0$ is a consequence of the assertions $p_n=0$, $p_{n+1}=0$ and $p_{n+2}=0$ taken jointly, and it is granted that $p_1=0$, $p_2=0$ and $p_3=0$. The conclusion is that $p_4=0$, $p_5=0 \ldots p_n=0$, when n is any positive integer.

But the term Mathematical Induction may have still wider application. The following general statement would include many special forms. Let $p_1=0$, $p_2=0$, $\ldots$ $p_r=0$ be a set of propositions, and let it be granted

(1) that between certain groups of these propositions relations exist of the type $\phi(p_a, p_b, p_c, \ldots p_h, p_k)=0$ in virtue of which we can infer $p_k=0$ if the "errors" $p_a, p_b \ldots p_h$ are all zero;

(2) that the relations ϕ can be placed in a certain order, say $\phi_1, \phi_2, \phi_3, \phi_4 \ldots$ such that each relation involves only one new "error" *i.e.* only one p which has not already occurred in the preceding relations, that one being p_k;

(3) that if $p_{a_1}, p_{b_1}, \ldots p_{h_1}, p_{k_1}$ be the errors involved in the first relation $\phi_1 = 0$, it is granted that $p_{a_1} = 0$, $p_{b_1} = 0$, $\ldots p_{h_1} = 0$.

It will then follow that all the propositions of the set under consideration must be admitted as true.

The most usual case is when the ϕ-relations are all of the same type, each referring to the same number of "errors." In this case the successive relations may be written
$\phi(p_0, p_1, p_2 \ldots p_r) = 0$, $\phi(p_1, p_2, p_3 \ldots p_{r+1}) = 0 \ldots \phi(p_s, p_{s+1}, p_{s+2}, \ldots p_{s+r}) = 0$;
and the propositions which must be granted as a basis for the induction, are $p_0 = 0$, $p_1 = 0$, $\ldots p_{r-1} = 0$, if we suppose that $p_0, p_1, p_2 \ldots$ are successive terms in a regular series of propositions.

A more general case arises when we have the relations

$$\phi(p_\alpha, p_\beta, p_\gamma \ldots p_\lambda) = 0\,;\ \phi(p_{\alpha+1}, p_{\beta+1}, \ldots p_{\lambda+1}) = 0\,;$$

$$\phi(p_{\alpha+2}, p_{\beta+2}, \ldots p_{\lambda+2}) = 0 \ldots \phi(p_{\alpha+s}, p_{\beta+s}, \ldots p_{\lambda+s}) = 0.$$

In this case the necessary "basis" of the induction will depend on the mutual relations of $\alpha, \beta, \gamma \ldots \lambda$.

If, for example, $\alpha, \beta, \ldots \lambda$ form an arithmetic progression of k terms with common difference d, a sufficient basis would be afforded by the $kd - 1$ propositions $p_1 = 0$, $p_2 = 0 \ldots p_{kd-1} = 0$.

The case in which there is a two dimensional array of propositions to be proved deserves special treatment. A simple case of this is the elementary theorem ${}^nC_r = \dfrac{n \,.\, (n-1)(n-2)\ldots(n-r+1)}{r\,!}$
which has to be proved for all positive integral values of n, and all positive integral values of r which are not greater than n.

From the identities ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$ and

$$\frac{(n+1)n(n-1)\ldots(n-r)}{r\,!} = \frac{n(n-1)\ldots(n-r+1)}{r\,!} + \frac{n(n-1)\ldots(n-r)}{r-1\,!}$$

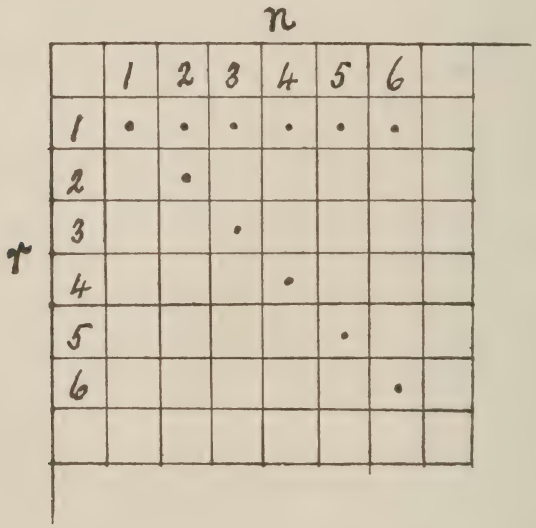
we deduce the relation $p_{n+1,\,r} - p_{n,\,r} - p_{n,\,r-1} = 0$ .....(4),

where $p_{n,\,r}$ stands for ${}^{n}C_{r} - \dfrac{n\,\overline{n-1}\ldots\overline{n-r+1}}{r\,!}$.

Here the ϕ-relation (4) implies not only that $p_{n+1,\,r} = 0$ follows from $p_{n,\,r} = 0$ together with $p_{n,\,r-1} = 0$, but that if *any* two of the errors are both zero, the third also must be zero.

The propositions required to start with in this case are infinite in number, but are all proved by a simpler mathematical induction, and for the present purpose may be assumed to be true. They are the propositions $p_{1,1} = 0$, $p_{2,2} = 0$, $p_{3,3} = 0 \ldots p_{m,m} = 0 \ldots$
and $p_{2,1} = 0$, $p_{3,1} = 0$, $p_{4,1} = 0 \ldots p_{m,1} = 0 \ldots$

To make it clear that the proposition $p_{n,\,r} = 0$ can be reached by induction for any significant values of n and r let us take a row-and-column table, as shewn in Fig. 1, where each place refers to



certain values of n and of r, and the dots indicate the values of n and r for which $p_{n,\,r} = 0$ is given.

The "graph" of the relation (4) is clearly of the form ⁚, and it is easy to see that by superposing it sufficiently often on the table-

spaces beginning with the first and second rows, proceeding with the second and third rows, and so on, we can reach any place to the right of the dotted diagonal of the table.

In order to have no blanks in the square array of the places in the table for this particular application it would be necessary to write $n+r=1$ instead of n in the expression for which $p_{n,\,r}$ stands, so that we should have $p_{n,\,r} \equiv {}^{n+r-1}C_r - \dfrac{(n+r-1)!}{(n-1)!\,r!}$ the relation $\phi = 0$ will still be $p_{n+1,\,r} - p_{n,\,r} - p_{n,\,r-1} = 0$, but there will now be a meaning for $p_{n,\,r}$ for any positive integral values of n and r.

The graph of ϕ is still of the form ⁚ ▪, and the necessary basis of admitted propositions will consist of those corresponding to all the places in the one row and in one column, *e.g.* in the first row and the first column.

In what immediately follows we will suppose that the array of places is square, and includes places corresponding to all positive integral values of n and of r.

The preceding example suggests a discussion of the "*pousto*" or graph of the basis of admitted propositions which will suffice for different forms of induction-graphs on a two-spread in order that complete induction by means of the corresponding ϕ-relation should be possible.

The simplest graphs will be those consisting of two places only. The graph ▪ ▪ requires as "pousto" no more than the first column of places, and if in $\phi(p_{n,\,r},\, p_{n,\,r+1}) = 0$, $p_{n,\,r} = 0$, and $p_{n,\,r+1}$ mutually imply one another, the first column may be replaced by one place taken arbitrarily from each row.

Similarly for the graph ⁚ a sufficient "pousto" would be one place in each column.

For the graph ▪ ▪ the first row and the first column would afford a minimum pousto; but for the graph ▪ ▪ the first row *or* the first column would suffice.

For the graph ▪ ∘ ▪ the first two columns would serve as pousto, for ▪ ∘ ∘ / ∘ ∘ ▪ the first row and the first two columns, for ∘ ∘ ▪ / ▪ ∘ ∘ the first row alone, *or* the first and second columns, and so on for other binomial graphs.

For $\begin{smallmatrix}\cdot & \cdot \\ \cdot & \circ\end{smallmatrix}$ the first row *or* the first column would suffice, while for $\begin{smallmatrix}\cdot & \circ \\ \cdot & \cdot\end{smallmatrix}$, for $\begin{smallmatrix}\circ & \cdot \\ \cdot & \cdot\end{smallmatrix}$ and $\begin{smallmatrix}\cdot & \cdot \\ \circ & \cdot\end{smallmatrix}$ both a row and a column would be required. There is thus a distinction between graphs which require only a row or a column for pousto, and those which require two rows, or a row and a column, or other combinations.

In general, if the graph of a ϕ can be contained in a rectangular block of places with μ rows and ν columns, a sufficient "pousto" will be afforded by $\mu - 1$ rows and $\nu - 1$ columns; but in some cases a smaller basis will suffice, some requiring rows only or columns only.

If we have two independent ϕ relations having graphs of fixed shape, then in many cases a finite number of places will serve as "pousto" on the graph-table. For example, any one place is a sufficient pousto for the simultaneous graphs $\begin{smallmatrix}\cdot \\ \cdot\end{smallmatrix}$ and $\begin{smallmatrix}\cdot & \cdot\end{smallmatrix}$, if the implication of the p's is mutual; and if otherwise, the place common to the first row and column is all that is required.

The graphical treatment of mathematical induction can be extended to cases where there are three or more independent integral parameters, by employing arrays of places of three or more dimensions, and similar remarks will apply to the nature of the "pousto" in various cases.

The problem of determining a ϕ-relation which will enable us to prove a general proposition involving one, or more than one, integral parameter is perhaps worthy of investigation. On this topic I content myself by offering one or two remarks.

Suppose the general proposition to be $p(n, r, q\ldots) = 0$. If $p(n, r, q\ldots) \equiv \alpha(n, r, q\ldots) - \alpha'(n, r, q\ldots)$, and if we can prove two related identities

$$A\alpha + B\beta + C\gamma + \ldots = 0$$
$$A\alpha' + B\beta' + C\gamma' + \ldots = 0$$

where α, β, $\gamma\ldots$ are cases of $\alpha(n, r, q)$ with certain related values of the parameters, and α', β', $\gamma'\ldots$ are the corresponding cases of $\alpha'(n, r, q)$, then we can deduce the identity

$$A(\alpha - \alpha') + B(\beta - \beta') + C(\gamma - \gamma') + \ldots = 0$$

or

$$Ap_{\alpha} + Bp_{\beta} + Cp_{\gamma} + \ldots = 0,$$

where p_{α} is the case of $p(n, r, q\ldots)$ corresponding to α, and so on. This is a ϕ-relation which may be used to effect an induction, provided a sufficient basis can be established. The proof of the formula for ${}^{n}C_{r}$ given above affords an illustration of this procedure, in the case when the number of parameters is two.

The general problem of this section would be: Given a theorem $p(n, r, q\ldots) = 0$ to be proved, to find a relation

$$\phi\{p(n_1 r_1 q_1\ldots), p(n_2 r_2 q_2\ldots), p(n_3, r_3, q_3)\ldots\} = 0$$

where n_1, r_1, $q_1\ldots n_2$, r_2, $q_2\ldots$, etc., are integral parameters having definite relations to one another. This problem would probably in most cases have an infinite number of solutions.

The converse problem is: Given the form of the relation ϕ, to find a proposition p which satisfies the relation. It also would probably have an indefinite number of different solutions.

On a certain Class of Linear Substitutions with Common Invariants, and an Associated Substitution of Order Four.

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Section I.—A Class of Linear Substitutions with Common Invariants.

§1. The determinant

$$D \equiv \begin{vmatrix} a_1, & a_2, & \dots & a_{n-1}, & a_n \\ a_n, & a_1, & \dots & a_{n-2}, & a_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ a_2, & a_3, & \dots & a_n, & a_1 \end{vmatrix},$$

each row of which contains the same n elements in the same cyclic order, with a_1 always in the leading diagonal, is the product of n linear factors,* which we shall write as follows:

$$k_\alpha \equiv \sum_{s-1}^{n} a_s \rho^{-(s-1)\alpha}, \ (\alpha = 1, 2, \dots n),$$

where ρ is any primitive n^{th} root of unity. That k_α is a factor of D is at once evident on multiplying the respective columns after the first by $\rho^{-\alpha}$, $\rho^{-2\alpha}$, $\dots$ $\rho^{-(n-1)\alpha}$, and the respective rows after the first by the same quantities in reversed order, and then adding all the rows (or columns) together.

Consider the linear substitution of which D is the determinant:

$$\left.\begin{aligned} x'_1 &= a_1x_1 + a_2x_2 + \dots + a_nx_n \\ x'_2 &= a_nx_1 + a_1x_2 + \dots + a_{n-1}x_n \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ x'_n &= a_2x_1 + a_3x_2 + \dots + a_1x_n \end{aligned}\right\} \dots\dots\dots\dots\dots(1).$$

* See Scott, *Determinants*, p. 81.

Form its *characteristic equation*

$$f(k) = 0$$

by putting

$$x'_\beta = kx_\beta, \ (\beta = 1, 2, \ldots n),$$

and eliminating the x_β. The roots of $f(k)$ are clearly the above k_α; and on substituting $k_\alpha x_\beta$ for x'_β in equations (1) we find that they are satisfied, provided

$$\frac{x_1}{\rho^{-\alpha}} = \frac{x_2}{\rho^{-2\alpha}} = \ldots = \frac{x_{n-1}}{\rho^{-(n-1)\alpha}} = \frac{x_n}{1}.$$

These values of $x_1, \ldots x_n$ define an invariant point* or *pole* P_α of the substitution. By varying α we obtain n poles $P_1, P_2, \ldots P_n$, forming a complete set of poles, with their coordinates *entirely independent of the coefficients* a_r. With the one restriction that D must not vanish, the a_r may have any real or complex values, and every substitution of the form (1) possesses the common system of invariant points $P_1, \ldots P_n$.

The corresponding linear invariants, or *invariant* $(n-2)$ *planes*, also possessed in common, are easily seen to be

$$\xi_\alpha \equiv \sum_{\beta=1}^{n} \rho^{\alpha\beta} x_\beta = 0, \ (\alpha = 1, 2, \ldots n) \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

§ 2. The result of successive applications of substitutions of the class (1), whether alike or different, is always another substitution of the class, the *multipliers* (or roots of the characteristic equation) for the resultant being the products of corresponding multipliers for the components. The substitution inverse to (1) is also of the same class.

There will be differences of *type* † within the class, according to the system of equalities among the multipliers. The minimum system of invariant points and $(n-2)$-planes, viz. the P_α and the ξ_α, belongs to all; but these will not exhaust the invariant points and $(n-2)$-planes of a substitution having one or more equalities among

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* *Linear Substitutions and their Invariants*, §2. *Proc. Edin. Math. Soc.*, Vol. XXX.

† *Ibid.* §9.

its multipliers. Such a substitution will have at least one *line* of poles, and at least one pencil of invariant $(n-2)$-planes.

Given the P_α (or the ξ_α) and the corresponding multipliers, we can build up the substitution uniquely. If the mutual ratios of the multipliers are r^{th} roots of unity, the substitution will be of order r. (*Ibid.* § 5.) We can thus construct substitutions of the class (1) of any order we please.

Section II.—The Associated Substitution of Order Four $S(\rho)$.

§ 3. Consider the substitution which transforms the frame of reference into the invariant frame of (1). Writing x' in place ξ in (2), and introducing the numerical factor $n^{-\frac{1}{2}}$, taken always with the *positive* sign, we express it in the form

$$x'_\alpha = n^{-\frac{1}{2}} \sum_{\beta=1}^{n}{}' \rho^{\alpha\beta} x_\beta, \ (\alpha = 1, 2, \ldots n) \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The introduction of the factor $n^{-\frac{1}{2}}$ reduces the modulus of the determinant of (3) to unity, as will appear shortly. We shall denote this substitution by $S(\rho)$, the left-hand member of its characteristic equation by $\Delta_n(\rho, k)$, and its determinant by $\Delta_n(\rho, 0)$ or simply $\Delta_n(\rho)$.

Thus, with $k = n^{-\frac{1}{2}}\kappa$,

$$\Delta_n(\rho, k) \equiv (n^{-\frac{1}{2}})^n \begin{vmatrix} \rho - \kappa, & \rho^2 & , \ldots\ldots & \rho^{n-1}, & , 1 \\ \rho & , \rho^4 - \kappa & , \ldots\ldots & \rho^{2(n-1)} & , 1 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ \rho^{n-1}, & \rho^{2(n-1)} - \kappa, & \ldots\ldots & \rho^{(n-1)^2} - \kappa, & 1 \\ 1 & , 1 & , \ldots\ldots & 1 & , 1 - \kappa \end{vmatrix}.$$

To obtain the substitution inverse to $S(\rho)$ multiply both sides of (3) by $\rho^{\gamma\alpha}$, and add for the n values of α. The coefficient of x_β on the right is

$$n^{-\frac{1}{2}} \sum_{\alpha=1}^{n} \rho^{\alpha(\beta+\gamma)},$$

which differs from zero only in the case of $\beta = n - \gamma$, and then takes the value $n^{\frac{1}{2}}$. Hence

$S^{-1}(\rho)$. $$x_{n-\gamma} = n^{-\frac{1}{2}} \sum_{\alpha=1}^{n} \rho^{\gamma\alpha} x'_\alpha, \ (\gamma = 1, 2, \ldots n), \ldots\ldots\ldots (4)$$

where x_0 is regarded as equivalent to x_n.

The only difference between (4) and (3) is that on the left of (4) the order of the first $(n-1)$ variables $x_1, \ldots x_{n-1}$ is reversed.

Now let x''_γ $(\gamma = 1, 2, \ldots n)$ be the result of two applications of $S(\rho)$. Then

$S^2(\rho)$.
$$x''_\gamma = n^{-\frac{1}{2}} \sum_{\beta=1}^{n} \rho^{\gamma\beta} x'_\beta \text{ by (3)},$$
$$= x_{n-\gamma} \text{ by (4)};$$

the original variables being restored, but, with the exception of the last, which remains unchanged, *reversed in order*. It follows that $S^4(\rho)$ is equivalent to the identical substitution, and that $S(\rho)$ *is of order four*. Further, since the multiplier with which the variables are restored is unity, we have

$$\Delta^4_n(\rho) = 1,$$
$$\Delta_n(\rho) = exp(\tfrac{1}{2}p\pi i), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

p having one of the values 0, 1, 2, 3.

§ 4. $S(\rho)$ being of order four, the roots of its characteristic equation must be fourth roots of unity (*Ibid*, § 5), whence

$$\Delta_n(\rho, k) \equiv (-)^n (k-1)^q (k+1)^r (k-i)^s (k+i)^t,$$

where
$$q + r + s + t = n. \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

In finding the indices, we shall confine ourselves to a specific value of ρ,

$$\rho = exp(2\pi i/n).$$

The more general case need not detain us, for other values of ρ lead to the same quantities x'_α in (3), only in a different order.

By multiplication of determinants we find for

$$\left.\begin{aligned} n = 2m+1,\ \Delta_n(\rho, k)\Delta_n(\rho, -k) &= -(k^2-1)^{m+1}(k^2+1)^m \\ = 2m, \qquad\qquad\qquad\quad &= \ (k^2-1)^{m+1}(k^2-1)^{m-1} \end{aligned}\right\} \ldots(7)$$

By a theorem of Gauss (see *Mathews*, *Theory of Numbers*, §184), writing

$$S = \sum_{s=0}^{a-1} exp(2s^2\pi i/n),$$

we have for

$$\begin{aligned} n \equiv 0 \pmod 4, \quad S &= (1+i)\sqrt{n} \\ \equiv 1 \qquad\quad\ = &\quad \sqrt{n} \\ \equiv 2 \qquad\quad\ = &\quad 0 \\ \equiv 3 \qquad\quad\ = &\quad i\sqrt{n}. \end{aligned}$$

Now the coefficient of k^{n-1} in $\Delta_n(\rho, k)$ is

$$(-)^{n-1}n^{-\frac{1}{2}}\mathrm{S};$$

and with $n = 2m+1$, $\Delta_n(\rho, k)$ must by (7) have the form

$$\Delta_n(\rho, k) = -(k-1)^{\alpha}(k+1)^{m+1-\alpha}(k-i)^{\beta}(k+i)^{m-\beta}.$$

Picking out the coefficient of the second-highest power of k, we have for

$$n \equiv 1 \pmod 4,\ (2\alpha - m - 1) + i(2\beta - m) = 1,\ \alpha = \tfrac{1}{2}m+1,\ \beta = \tfrac{1}{2}m;$$
$$n \equiv 3 \qquad (2\alpha - m - 1) + i(2\beta - m) = i,\ \alpha = \tfrac{1}{2}(m+1) = \beta.$$

Again with $n = 2m$, $\Delta_n(\rho, k)$ is of form

$$\Delta_n(\rho, k) = (k-1)^{\alpha}(k+1)^{m+1-\alpha}(k-i)^{\beta}(k+i)^{m-1-\beta},$$

whence for

$$n \equiv 0 \pmod 4,\ \alpha = \tfrac{1}{2}m+1,\ \beta = \tfrac{1}{2}m;$$
$$\equiv 2 \qquad \alpha = \tfrac{1}{2}(m+1),\ \beta = \tfrac{1}{2}(m-1).$$

The following table summarises these results, and also gives the values of p in (5):

n	$\Delta_n(\rho, k)$	$p \equiv (\mathrm{mod}\ 4)$
$4s$	$(k-1)^{s+1}(k+1)^{s}(k-i)^{s}(k+i)^{s-1}$	$2s+1$
$4s+1$	$-(k-1)^{s+1}(k+1)^{s}(k-i)^{s}(k+i)^{s}$	$2s$
$4s+2$	$(k-1)^{s+1}(k+1)^{s+1}(k-i)^{s}(k+i)^{s}$	$2s+2$
$4s+3$	$-(k-1)^{s+1}(k+1)^{s+1}(k-i)^{s+1}(k+i)^{s}$	$2s+3$

§5. Taking q, r, s, t as in (6), the canonical form of $\mathrm{S}(\rho)$ is clearly the following:

$$\xi'_{\alpha} = \xi_{\alpha}, \qquad (\alpha = 1, 2, \ldots\ldots q),$$
$$\xi'_{\beta} = -\xi_{\beta}, \quad (\beta = q+1, \ldots\ldots q+r),$$
$$\xi'_{\gamma} = i\xi_{\gamma}, \quad (\gamma = q+r+1, \ldots\ldots q+r+s),$$
$$\xi'_{\delta} = -i\xi_{\delta}, \quad (\delta = q+r+s+1, \ldots\ldots n),$$

the ξ being a set, not in general unique, of linear invariants of $\mathrm{S}(\rho)$. It remains to express the ξ in terms of the original co-ordinates x.

A separation can be made between the ξ with real and those with imaginary multipliers; the former appearing as the invariants of a new substitution in $(q+r)$ variables, and the latter in the same relation to another substitution in $(s+t)$ variables. The reduction shows that the ξ are in all cases *real* linear functions of the x; thus the linear invariants, and therefore also the poles of $S(\rho)$ are all real.

An invariant of $S(\rho)$ with real multiplier (± 1) is an invariant of $S^2(\rho)$ with multiplier $+1$, and therefore by (4) is unaltered by interchange throughout of x_α, $x_{n-\alpha}$, $(\alpha = 1, 2, \ldots, n-1)$. An invariant of $S(\rho)$ with imaginary multiplier $(\pm i)$ is an invariant of $S^2(\rho)$ with multiplier -1, and therefore simply changes sign on interchange throughout of x_α, $x_{n-\alpha}$.

Write

$$x_\alpha + x_{n-\alpha} = y_\alpha,$$

$$x_\alpha - x_{n-\alpha} = z_\alpha,$$

where α takes all integral values from 1 to m or $m-1$ according as $n = 2m+1$ or $2m$. In the latter case write $x_m = y_m$, and in both cases $x_n = y_{m+1}$. Then the invariants with real multipliers are linear functions of the y, those with imaginary multipliers of the z.

But putting

$$\left.\begin{aligned} \rho^{\alpha} + \rho^{-\alpha} &\equiv 2\cos(2\pi\alpha/n) = \sigma_\alpha \\ -i(\rho^{\alpha} - \rho^{-\alpha}) &\equiv 2\sin(2\pi\alpha/n) = \tau_\alpha \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

and taking

(i) $n = 2m+1$, we have from (3),

$$\left.\begin{aligned} \sqrt{n}\cdot y'_\alpha &= \sum_{\beta=1}^{m} \sigma_{\alpha\beta} y_\beta + 2y_{m+1}, \quad (\alpha = 1, 2 \ \ldots \ m) \\ \sqrt{n}\cdot y'_{m+1} &= \sum_{\beta=1}^{m} y_\beta + y_{m+1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (9)$$

and

$$\sqrt{n}\cdot z'_\alpha = \sum_{\beta=1}^{m} i\tau_{\alpha\beta} z_\beta, \ (\alpha = 1, 2, \ \ldots \ m) \ldots\ldots\ldots\ldots\ldots (10)$$

The former is a substitution of order 2 in the $(m+1)$ variables $y_1 \ldots y_{m+1}$, of which the invariants, when expressed in terms of the x, are those of (3) with real multipliers, and have real coefficients throughout. The invariants of (10) are likewise identical with

those of (3) which have imaginary multipliers; and since, if we omit the i on the right-hand side of (10), the only adjustment required is to alter the multipliers from $\pm i$ to ± 1 respectively, the invariants themselves remaining unchanged, it is clear that they are *real* functions of the z, and hence also of the x.

(ii) Similarly for $n = 2m$. Equations (9) are replaced by

$$\left.\begin{aligned}
\sqrt{n}\,.\,y'_\alpha &= \sum_{\beta=1}^{m-1} \sigma_{\alpha\beta} y_\beta + (-)^\alpha 2y_m + 2y_{m+1}, \ (\alpha = 1, \ldots m-1) \\
\sqrt{n}\,.\,y'_m &= \sum_{\beta=1}^{m-1} (-)^\beta y_\beta + (-)^m\ y_m + \ y_{m+1}, \\
\sqrt{n}\,.\,y'_{m+1} &= \sum_{\beta=1}^{m-1} \qquad y_\beta + \qquad\quad y_m + \ y_{m+1}
\end{aligned}\right\} \quad \ldots\ldots(11)$$

and (10) by

$$\sqrt{n}\,.\,z'_\alpha = \sum_{\beta=1}^{m-1} i\tau_{\alpha\beta} z_\beta, \ (\alpha = 1, \ \ldots \ m-1) \ \ldots\ldots\ldots\ldots(12)$$

On comparing the forms assumed by the characteristic equation in (9) to (12) with those of the table of § 4, identities are obtained, of which the following may be noted:

With $n = 2m+1 = 4s+1$, the σ, τ having the values defined in (8),

$$\begin{vmatrix}
\sigma_1 - \kappa, & \sigma_2 & , \ \ldots\ldots & \sigma_m & , & 2 \\
\sigma_2 & , & \sigma_4 - \kappa, \ \ldots\ldots & \sigma_{2m} & , & 2 \\
\ldots\ldots & & & & & \\
\sigma_m & , & \sigma_{2m} , \ \ldots\ldots & \sigma_{m^2} - \kappa, & & 2 \\
1 & , & 1 , \ \ldots\ldots & 1 & , & 1 - \kappa
\end{vmatrix} = -(\kappa - \sqrt{n})\,(\kappa^2 - n)^s,$$

$$\begin{vmatrix}
\tau_1 - \kappa, & \tau_2 & , \ \ldots\ldots & \tau_m \\
\tau_2 & , & \tau_4 - \kappa, \ \ldots\ldots & \tau_{2m} \\
\ldots\ldots & & & \\
\tau_m & , & \tau_{2m} , \ \ldots\ldots & \tau_{m^2} - \kappa
\end{vmatrix} = (\kappa^2 + n)^s.$$

The invariants ξ obtained by the help of (9)–(12) as far as $n = 8$, are tabulated below. Only those with multipliers $+1$, $+i$ are written down; those with multipliers -1, $-i$ are deduced therefrom by changing the sign of $\sqrt{n}$ wherever it occurs explicitly. In the cases of two or more invariants with the same

multiplier, those given are not unique, and any linear functions of them would serve equally well. The form

$$\sum_{\alpha=1}^{n} x_\alpha + \sqrt{n} \cdot x_n$$

which appears first in each set, is at once seen to be invariant from (3). It is to be remembered that the σ, τ depend on the value of n.

Since the determinant of (3) is symmetrical, the pole corresponding to any invariant

$$\sum_{\alpha=1}^{n} p_\alpha x_\alpha$$

can be immediately deduced: it is in fact the point with coordinates $(p_1, p_2 \ldots p_n)$. The extension to the case of equal multipliers is easily made.

Relations come to light in the process of finding the forms, which are exemplified in the case of $n=7$. For this case the substitution (9) will have as an invariant with multiplier $+1$ the function

$$\sum_{\alpha=1}^{4} p_\alpha y_\alpha$$

provided

$$\begin{aligned}
-(\sqrt{7}-\sigma_1)p_1 + \quad & \sigma_2 p_2 + \quad & \sigma_3 p_3 + \quad & p_4 = 0 \\
\sigma_2 p_1 - (\sqrt{7}-\sigma_3)p_2 + \quad & & \sigma_1 p_3 + \quad & p_4 = 0 \\
\sigma_3 p_1 + \quad & \sigma_1 p_2 - (\sqrt{7}-\sigma_2)\, p_3 + \quad & & p_4 = 0 \\
2p_1 + \quad & 2\, p_2 + \quad & 2\, p_3 - (\sqrt{7}-1)\, p_4 = 0.
\end{aligned}$$

Since there are two invariants of this kind, only two of these four equations can be independent; in other words, each first minor of the determinant of the coefficients is zero. This can be verified in virtue of the relations

$$\begin{aligned}
\sigma_1 + \sigma_2 \quad + \sigma_3 \quad &= -1, \\
\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2 &= -2\,;
\end{aligned}$$

and the statement remains true when the sign of the radical is changed.

Similarly the conditions that

$$\sum_{\alpha=1}^{3} p_\alpha z_\alpha$$

be an invariant of (10) for this case, with multiplier $+i$, are

$$
\begin{aligned}
-(\sqrt{7}-\tau_1)p_1 + \quad & \tau_2 p_2 + \quad & \tau_3 p_3 = 0 \\
\tau_2 p_1 - (\sqrt{7}+\tau_3)p_2 - \quad & & \tau_1 p_3 = 0 \\
\tau_3 p_1 - \quad & \tau_1 p_2 - (\sqrt{7}-\tau_2)p_3 = 0,
\end{aligned}
$$

only one of which can be independent. This is evident on remembering that

$$\tau_1 + \tau_2 - \tau_3 = \sqrt{7},$$
$$\tau_2\tau_3 + \tau_3\tau_1 - \tau_1\tau_2 = 0,$$

but is no longer true when the sign of the radical in the three conditions is changed.

§ 6. The application of the substitution (3) to the transformation of algebraic equations may be indicated. The equations

$$\phi_n(x) \equiv \prod_{\alpha=1}^{n} (x - x_\alpha) = 0,$$

$$\psi_n(x) \equiv \prod_{\alpha=1}^{n} (x - x'_\alpha) = 0$$

are reciprocally related, the roots of each being derived from those of the other by the same linear substitution.

Consider the effect of increasing the roots of $\phi_n(x)$ by the constant quantity λ. It is clear that all the roots of $\psi_n(x)$ will remain unchanged, except x'_n, which will be increased by $\sqrt{n}\,.\,\lambda$. If therefore we choose $\lambda = -n^{-1}\sum_{\alpha=1}^{n} x_\alpha$, two things will happen:

(i) the term in x^{n-1} will drop out from $\phi_n(x)$;

(ii) the new x'_n will vanish, and $\psi_n(x)$ be replaced by the function of degree one lower

$$\psi_{n-1}(x) \equiv \prod_{\alpha=1}^{n=1} (x - x'_\alpha).$$

Were it possible to obtain the x'_α in terms of the coefficients of $\phi_n(x)$, $\phi_n(x)$ could be solved. In the case of the cubic

$$\phi_3(x) \equiv x^3 + 3px + q = 0,$$

(3) reduces, after the change indicated, to the equations

$$x_1' = 3^{-\frac{1}{2}}(\omega x_1 + \omega^2 x_2 + x_3),$$
$$x_2' = 3^{-\frac{1}{2}}(\omega^2 x_1 + \omega x_2 + x_3),$$
$$0 = x_3' = 3^{-\frac{1}{2}}(x_1 + x_2 + x_3);$$

TABLE OF LINEAR INVARIANTS OF $S(\rho)$.

n	mult + 1	mult + i
2	$\Sigma x + \sqrt{2} \cdot x_2$	none
3	$\Sigma x + \sqrt{3} \cdot x_3$	$x_1 - x_2$
4	$\Sigma x + \sqrt{4} \cdot x_4$ $x_2 + x_4$	$x_1 - x_3$
5	$\Sigma x + \sqrt{5} \cdot x_5$ $(x_1 + x_4) - (x_2 + x_3)$	$(\tau_1 + \sqrt{5})(x_1 - x_4) + \tau_2(x_2 - x_3)$
6	$\Sigma x + \sqrt{6} \cdot x_6$ $(2 + \sqrt{6})(x_1 + x_5) - \sqrt{6}(x_2 + x_4) - 4x_3$	$(\tau_1 + \sqrt{6})(x_1 - x_5) + \tau_2(x_2 - x_4)$ (here $\tau_1 = \tau_2 = + \sqrt{3}$)
7	$\Sigma x + \sqrt{7} \cdot x_7$ $(\sigma_1 - \sigma_2 + \sqrt{7})(x_1 + x_6) + (\sigma_2 - \sigma_3 - \sqrt{7})(x_2 + x_5) + (\sigma_3 - \sigma_1)(x_3 + x_4)$	$\tau_2(x_1 - x_6) - (\tau_1 - \sqrt{7})(x_2 - x_5)$ $(x_1 - x_6) + (x_2 - x_5) - (x_3 - x_4)$
8	$\Sigma x + \sqrt{8} \cdot x_8$ $4(x_1 + x_7) - \sqrt{8} \cdot x_4 + \sqrt{8} \cdot x_8$ $(x_1 + x_7) - (x_3 + x_5)$	$4(x_1 - x_7) + \sqrt{8}(x_2 - x_6)$ $(x_1 - x_7) + (x_3 - x_5)$

whence

$$x_1' x_2' = -\Sigma x_2 x_3 = -3p,$$

$$x_1'^3 + x_2'^3 = 3\sqrt{3} \cdot x_1 x_2 x_3 = -3\sqrt{3} \cdot q,$$

leading to the usual solution. In the case of the biquadratic also the method leads to Euler's reducing cubic.

Section III.—Triangles in Multiple Perspective. Geometrical Interpretation of Substitution (1) for $n = 3$.

§ 7. The triangle $X_1X_2X_3$ is said to be in perspective with $Y_\alpha Y_\beta Y_\gamma$, the order of the vertices being significant, when X_1Y_α, X_2Y_β, X_3Y_γ meet in a point. Thus $X_1X_2X_3$ may be in perspective with another triangle in either of six different ways, according to the order of the vertices. Let the vertices, referred to $X_1X_2X_3$ as triangle of reference in any system of homogeneous point-coordinates, be

$$Y_r\,(x_{r1},\ x_{r2},\ x_{r3}),\ (r = 1,\ 2,\ 3)\ ;$$

then the six cases, with their conditions, fall into two sets of three as follows: $X_1X_2X_3$ is in perspective with

(i)	$Y_1Y_2Y_3$	provided	$x_{12}x_{23}x_{31} = x_{13}x_{21}x_{32}$	
(ii)	$Y_2Y_3Y_1$	,,	$x_{11}x_{22}x_{33} = x_{12}x_{23}x_{31}$	(A)
(iii)	$Y_3Y_1Y_2$	,,	$x_{13}x_{21}x_{32} = x_{11}x_{22}x_{33}$	
(iv)	$Y_1Y_3Y_2$	,,	$x_{22}x_{31}x_{13} = x_{33}x_{12}x_{21}$	
(v)	$Y_3Y_2Y_1$	,,	$x_{33}x_{12}x_{21} = x_{11}x_{23}x_{32}$	(B)
(vi)	$Y_2Y_1Y_3$	,,	$x_{11}x_{23}x_{32} = x_{22}x_{31}x_{13}$	

The poles of perspective will be called $Z_r (r = 1, 2, \ldots 6)$. Sets (A), (B) are distinguished by the cyclic order of the vertices $Y_1Y_2Y_3$. In each set any two of the conditions involve the third. It is thus clear that

(α) *triple* perspective will occur when two conditions of one set are fulfilled, and therefore two of the vertices (say Y_2Y_3) may be taken at random. The three poles of perspective are clearly the vertices of a third triangle in triple perspective with each of the other two, the poles for each pair being the vertices of the remaining one ;

(β) *quadruple* perspective occurs when three conditions, not all of one set, are fulfilled. The construction of such a case

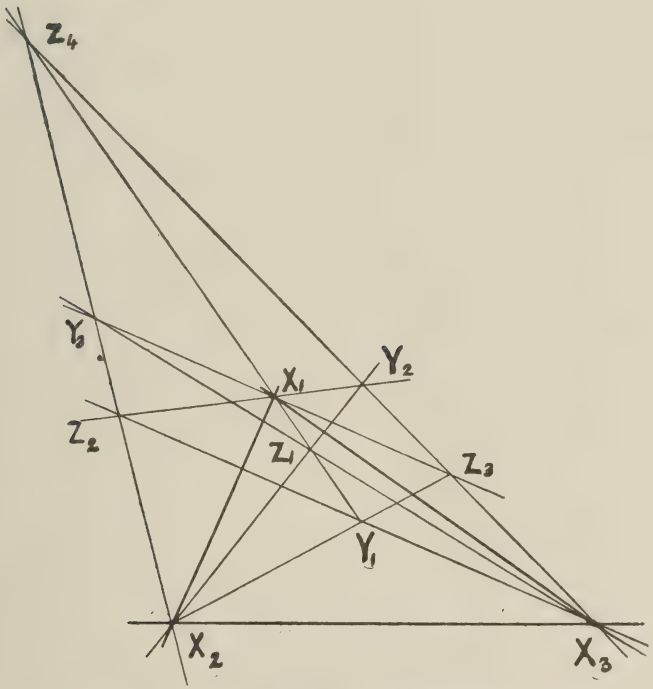
is a simple and instructive exercise in homogeneous co-ordinates.

One vertex (say Y_3) may be taken at random, and the system of coordinates so chosen that it becomes the point (1, 1, 1). Thus if cases (i)–(iv) hold, we have for the coordinates of the other vertices

$$x_{12}x_{23} = x_{13}x_{21} = x_{11}x_{22},$$
$$x_{13}x_{22} = x_{12}x_{21},$$

whence $$\frac{x_{12}}{x_{13}} = \frac{x_{21}}{x_{23}} = \frac{x_{22}}{x_{21}} = \frac{x_{13}}{x_{11}} = \alpha, \text{ say,}$$

and Y_1, Y_2 are given by $Y_1(1, \alpha^2, \alpha)$, $Y_2(\alpha, \alpha^2, 1)$, α being arbitrary.



Triangles $X_1X_2X_3$, $Y_1Y_2Y_3$ *in quadruple perspective.*

[The figure is given for an arbitrary position of Y_3, and $\alpha = -2$. The collinearity of $Z_1Z_2Z_3$ is a coincidence, due to the value of α

chosen. For it is easily found that the coordinates of the poles are

$$Z_1(\alpha, \alpha, 1),\ Z_2(1, \alpha^2, 1),\ Z_3(1, \alpha, \alpha),\ Z_4(1, \alpha, 1)\ ;$$

and the determinant of the coordinates of the first three $= \alpha(\alpha - 1)^2(\alpha + 2)$, vanishing for $\alpha = -2$.]

(γ) *Sextuple* perspective occurs on the fulfilment of all six conditions, but, there being only four independent, one vertex Y_3 can still be taken at random. The conditions yield

$$\frac{x_{11}}{x_{12}} = \frac{x_{12}}{x_{13}} = \frac{x_{13}}{x_{11}} = \frac{x_{22}}{x_{21}} = \frac{x_{23}}{x_{22}} = \frac{x_{21}}{x_{23}}$$
$$= \text{(clearly) } 1, \omega, \text{ or } \omega^2.$$

Unity is rejected as reducing $Y_1Y_2Y_3$ to a point. We take the value ω, and obtain the triangle $Y_1(\omega^2, \omega, 1)$, $Y_2(\omega, \omega^2, 1)$, $Y_3(1, 1, 1)$; the value ω^2 would simply interchange Y_1, Y_2. Thus *there is only one triangle in sextuple perspective with the triangle of reference, and with a vertex at an arbitrary point. If that point is the real point* (1, 1, 1), *the other two vertices are imaginary points on the real line*

$$x_1 + x_2 + x_3 = 0.$$

§ 8. Consider the substitution (1) for $n = 3$, applied to the system of coordinates here in use. It is evident that $Y_1Y_2Y_3$ is coincident with the invariant triangle of the substitution. We shall therefore revert to our previous notation and call it the triangle $P_1P_2P_3$. The six poles, in the order of § 7, are

(i) $Q_3(1, 1, \omega^2)$, (iv) $R_3(1, 1, \omega)$,
(ii) $Q_2(1, \omega^2, 1)$, (v) $R_2(1, \omega, 1)$,
(iii) $Q_1(\omega^2, 1, 1)$, (vi) $R_1(\omega, 1, 1)$.

To pass from one system of homogeneous coordinates to another, with the same triangle of reference, we simply alter the coordinates of each point in definite ratios, so that, *e.g.* the old coordinates of a point having been x_1, x_2, x_3, its new ones are $(\alpha x_1, \beta x_2, \gamma x_3)$, α, β, γ being constant for all points. Thus the triangle $T_1(\omega^2\alpha, \omega\beta, \gamma)$ $T_2(\omega\alpha, \omega^2\beta, \gamma)$ $T_3(\alpha, \beta, \gamma)$, whatever the system of coordinates, is in sextuple perspective with the triangle of reference, and can be made to coincide with any triangle in this relation, by giving suitable values to $\alpha : \beta : \gamma$. Now the two sets

of multipliers $\alpha : \beta : \gamma = \omega^2 : 1 : 1$ and $\omega : 1 : 1$ respectively change the points

$$P_1P_2P_3 \; Q_1Q_2Q_3 \; R_1R_2R_3$$

into $\quad Q_3Q_2Q_1 \; R_1R_3R_2 \; P_3P_1P_2,$

and $\quad R_2R_3R_1 \; P_3P_2P_1 \; Q_1Q_3Q_2.$

Hence each of the triangles $P_1P_2P_3$, $Q_1Q_2Q_3$, $R_1R_2R_3$ is in sextuple perspective with the triangle of reference, and the poles of perspective in each case are the vertices of the remaining two.

§ 9. The relation of $P_1P_2P_3$ to $X_1X_2X_3$ may be expressed by another geometrical analogy. A point and a line are called *harmonic pole and polar* with respect to a triangle in their plane when the line, and the joins of the point to the vertices, divide the three sides harmonically. If each vertex and opposite side of one triangle are harmonic pole and polar with respect to another, the former may be called *self-polar* with respect to the latter; and, as it easily follows from the conditions obtained below that the relation is reciprocal, the two triangles may be called *conjugate.*

To find the conditions that the three points

$$(x_{\alpha 1}, \; x_{\alpha 2}, \; x_{\alpha 3}), \; (\alpha = 1, \; 2, \; 3),$$

form a triangle self-polar with respect to the triangle of reference.

Denoting by $X_{\alpha\beta}$ the cofactor of $x_{\alpha\beta}$ in the determinant Δ of the coefficients, and expressing that each point is the harmonic pole of the line joining the other two, we obtain the nine relations

$$x_{\alpha\beta} X_{\alpha\beta} = \text{const.} = \tfrac{1}{3}\Delta \, . \, (\alpha, \; \beta = 1, \; 2, \; 3).$$

The two relations

$$x_{11}X_{11} = x_{12}X_{12}, \;\; x_{33}X_{33} = x_{23}X_{23}$$

may be written respectively

$$x_{33}(x_{11}x_{22} + x_{12}x_{21}) = x_{23}(x_{12}x_{31} + x_{11}x_{32}),$$

$$x_{33}(x_{11}x_{22} - x_{12}x_{21}) = x_{23}(x_{12}x_{31} - x_{11}x_{32}),$$

whence, and by symmetry,

$$x_{11}x_{22}x_{33} = x_{23}x_{31}x_{12} = x_{32}x_{13}x_{21},$$

$$x_{11}x_{23}x_{32} = x_{22}x_{31}x_{13} = x_{33}x_{12}x_{21}.$$

These conditions, which satisfy all the nine relations, are exactly those for sextuple perspective (§ 7). Thus *triangles in sextuple perspective are conjugate, and* vice versa.

The triangle $P_1P_2P_3$, being uniquely determined as soon as the system of coordinates is specified, might be fitly called *the principal related triangle* for the system.

§ 10. The substitution (1) becomes for $n = 3$,

$$\left.\begin{aligned} x_1' &= a_1x_1 + a_2x_2 + a_3x_3 \\ x_2' &= a_3x_1 + a_1x_2 + a_2x_3 \\ x_3' &= a_2x_1 + a_3x_2 + a_1x_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Taking x_1, x_2, x_3 as homogeneous point-coordinates, we must distinguish two possible interpretations. First, it might mean that the *point* whose coordinates are x_1, x_2, x_3 has to be shifted to (x_1', x_2', x_3'); thus (1, 0, 0) would be shifted to (a_1, a_3, a_2), etc. Secondly, it might mean that the *line* whose equation is

$$l_1x_1 + l_2x_2 + l_3x_3 = 0$$

is to be shifted into the position of the line

$$l_1x_1' + l_2x_2' + l_3x_3'$$

$$\equiv (a_1l_1 + a_3l_2 + a_2l_3)x_1 + (a_2l_1 + a_1l_2 + a_3l_3)x_2 + (a_3l_1 + a_2l_2 + a_1l_3)x_3 = 0\,;$$

so that $x_1 = 0$ would become

$$a_1x_1 + a_2x_2 + a_3x_3 = 0,$$

and so on.

If in (13) we replace a_1, a_2, a_3 respectively by A_1, A_3, A_2 the cofactors of a_1, a_3, a_2 in the determinant of the coefficients, the *first* interpretation of the new substitution is geometrically equivalent to the *second* interpretation of (13). We can therefore afford to ignore the second interpretation as virtually involved in the first. It is also easy to see that the geometrical effects of (13) in its two interpretations are mutually inverse.

§ 11. With the first interpretation, then, (13) shifts $X_1X_2X_3$ into $Y_1(a_1, a_3, a_2)$, $Y_3(a_2, a_1, a_3)$, $Y_2(a_3, a_2, a_1)$. These coordinates satisfy equations (A) of § 7; hence *every triangle into which* $X_1X_2X_3$ *is changed by* (13) *is in triple perspective with* $X_1X_2X_3$ *itself.*

The poles are found to be

$$Z_1(a_1^{-1}, a_2^{-1}, a_3^{-1}),\ Z_2(a_3^{-1}, a_1^{-1}, a_2^{-1}),\ Z_3(a_2^{-1}, a_3^{-1}, a_1^{-1}),$$

satisfying equations (B) of § 7, and therefore forming a triangle in triple perspective with $X_1X_2X_3$, but in reversed cyclic order.

The condition that Y_1, Y_2, Y_3 should satisfy (iv) in addition to (i)–(iii) is

$$a_2^3 = a_3^3\ ;$$

thus the equality of two coefficients in the substitution involves quadruple perspective between $X_1X_2X_3$ and every triangle obtained from it by the substitution; and the same holds if two coefficients have the ratio ω or ω^2, though the triangle is then imaginary. Whatever the system of point-coordinates and the value of α, the triangle $(\alpha, 1, 1)$, $(1, 1, \alpha)$, $(1, \alpha, 1)$ is in quadruple perspective in ways (i) to (iv) with $X_1X_2X_3$.

§ 12. The invariant points and lines of (13), forming the *invariant triangle*, are

$$\left.\begin{array}{ll} P_1(\omega^2, \omega, 1), & \sqrt{3}\,.\,\xi_1 = \omega x_1 + \omega^2 x_2 + x_3 \\ P_2(\omega, \omega^2, 1), & \sqrt{3}\,.\,\xi_2 = \omega^2 x_1 + \omega x_2 + x_3 \\ P_3(1, 1, 1), & \sqrt{3}\,.\,\xi_3 = x_1 + x_2 + x_3 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (14)$$

This triangle is what we have called (§9) *the principal related triangle* for the particular system of point-coordinates in use. It is in sextuple perspective with, and conjugate to, the triangle of reference (§§ 8, 9); and not only so, but, as perspective and harmonic properties are unaltered by a linear transformation, it *stands in the same relations to every triangle obtainable by* (13) *from the triangle of reference.* When we say "by (13)," we mean by every substitution of the *form* (13), the result being independent of the coefficients of (13).

§ 13. The geometrical interpretation of $S(\rho)$ for $n = 3$, or of $S(\omega)$, is plain: it changes the triangle of reference $X_1X_2X_3$ into the *principal related triangle* $P_1P_2P_3$ for the coordinates in use, the order of the vertices being significant. $S(\omega^2)$ will change $X_1X_2X_3$ into $P_2P_1P_3$, and substitutions of like form will change $X_1X_2X_3$ into each of the other four ways in which $P_1P_2P_3$ may be named. Each pair of substitutions, such as $S(\omega)$ and $S(\omega^2)$, gives *one* invariant triangle, and the nine vertices are as follows, the three

last deduced from the table of § 5, the others by cyclic interchange :—

$S_{11}(0, 1, -1)$, $S_{12}(\sqrt{3}+1, 1, 1)$, $S_{13}(-\sqrt{3}+1, 1, 1)$;

$S_{21}(1, -\sqrt{3}+1, 1)$, $S_{22}(-1, 0, 1)$, $S_{23}(1, \sqrt{3}+1, 1)$;

$S_{31}(1, 1, \sqrt{3}+1)$, $S_{32}(1, 1, -\sqrt{3}+1)$, $S_{33}((1, -1, 0)$.

They are all real, the three diagonal ones lying in a straight line, and the others, in two sets of three, forming triangles in triple perspective with $X_1X_2X_3$.

Under repetitions of $S(\omega)$, $X_1X_2X_3$ passes through the successive stages $P_1P_2P_3$, $X_2X_1X_3$, $P_2P_1P_3$, and so back to $X_1X_2X_3$, the points S_{31}, S_{32}, S_{33} remaining fixed ; and similarly in the other cases.*

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* Since writing the above, I have found the following references to triangles in multiple perspective :—

Homologous Triangles, by Thomas Muir, M.A. [*Mess. Math.* 2 (1873)]. "Triply homologous" triangles are discussed, the vertices being taken in the same *cyclic* order throughout.

Perspective Dreiecke und Tetraeder, by Edmund Hess [*Math. Annalen* 28 (1887)]. Here sextuply perspective triangles are introduced, the standpoint being geometrical, and the application different from that of the present paper.

Integral Equations and the Determination of Green's Functions in the Theory of Potential.

By Professor H. S. Carslaw.

(*Received 20th May 1912. Read 9th December 1912*).

In the Theory of Potential the term Green's Function, used in a slightly different sense by Maxwell, now denotes a function associated with a closed surface S, with the following properties:—

(i) In the interior of S, it satisfies $\nabla^2 V = 0$.

(ii) At the boundary of S, it vanishes.

(iii) In the interior of S, it is finite and continuous, as also its first and second derivatives, except at the point (x_1, y_1, z_1).

(iv) At the point (x_1, y_1, z_1), it becomes infinite as $\frac{1}{4\pi r}$, when r tends to zero, r being the distance from (x, y, z) to (x_1, y_1, z_1).

In other branches of Applied Mathematics similar functions have been introduced and called by the same name. These have become of increasing importance as kernels of Homogeneous Integral Equations. In the applications of Integral Equations in Mathematical Physics they occupy a prominent place.*

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* Cf. Kneser, *Die Integralgleichungen und ihre Anwendungen in der Mathematischen Physik* (*Braunschweig*, 1911).

We shall refer to Kneser's book in the following pages as Kneser, *Integralgleichungen.*

Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, Göttinger Nachrichten, Math.-Phys. Klasse*, 1904.

Stekloff, *Théorie générale des fonctions fondamentales, Ann. de la Fac. des Sc. de Toulouse* (2), T. 6, p. 351, 1904.

§ 2. Let (x_1, y_1, z_1) be the coordinates of the point P at which the Green's Function is to be infinite as $\frac{1}{4\pi r}$.

Let (x, y, z) be the coordinates of any point Q in the region bounded by S.

Then the Green's Function at Q will depend upon (x, y, z) and (x_1, y_1, z_1). Following Kneser,* we shall denote it by K(0, 1), and refer to the points P and Q as (0) and (1) respectively.

Thus we have, inside S,

$$\nabla^2 K(0, 1) = 0,$$

and

$$K(0, 1) = \frac{1}{4\pi r} + M(0, 1),$$

where M(0, 1) is a finite and continuous function of (0, 1) in the given region, its first and second differential coefficients with respect to x, y and z being also finite and continuous there.

Further, $K(0, 1) = 0$, on the surface S.

A simple application of Green's Theorem shows that K(0, 1) is a symmetrical function of (x, y, z) and (x_1, y_1, z_1).

In other words, we have

$$K(0, 1) = K(1, 0).$$

§ 3. Now let ψ be any solution of

$$\nabla^2\psi + \lambda\psi = 0,$$

which vanishes on the surface S, and with its first and second derivatives, is finite and continuous in the region bounded by S.

Applying Green's Theorem to the region between S and a small sphere Σ, whose centre is at the point (1), we have

$$\iiint \{\psi \nabla^2 K(0, 1) - K(0, 1)\nabla^2\psi\} dv + \iint \left\{\psi \frac{\partial}{\partial n} K(0, 1) - K(0, 1)\frac{\partial}{\partial n}\psi\right\} d\Sigma = 0,$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal to Σ, and dv the element of volume at the point (0).

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* Cf. Kneser, *Integralgleichungen*, p. 127.

When the radius of Σ tends to zero, we obtain the known result

$$\psi(1) = \lambda \iiint \mathrm{K}(0, 1)\, \psi(0) dv, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where $\psi(0)$ stands for $\psi(x, y, z)$ and $\psi(1)$ for $\psi(x_1, y_1, z_1)$.

Therefore the functions ψ are *characteristic functions* (*Eigenfunktionen*), and the numbers λ are *characteristic numbers* (*Eigenwerte*) of a Homogeneous Integral Equation with K(0, 1) as its *symmetrical kernel.**

§ 4. These characteristic functions are supposed to be *normalised*, so that

$$\iiint \psi^2 dv = 1,$$

the integration being taken through the interior of S.

If we assume that K(0, 1) can be expanded in an infinite series

$$\mathrm{A}_1\psi_1 + \mathrm{A}_2\psi_2 + \ldots,$$

which can be integrated term by term, on multiplying by ψ_n and integrating, we find that

$$\mathrm{A}_n \iiint \psi_n^2 dv = \iiint \mathrm{K}(0, 1)\psi_n(0) dv.$$

Therefore with this hypothesis we would have

$$\mathrm{A}_n = \iiint \mathrm{K}(0, 1)\, \psi_n(0) dv,$$

and

$$\mathrm{K}(0, 1) = \frac{\psi_1(0)\, \psi_1(1)}{\lambda_1} + \frac{\psi_2(0)\, \psi_2(1)}{\lambda_2} + \ldots\ldots\ldots\ldots\ldots (2)$$

We are thus led to enquire whether the series in (2) really converges and represents the function K(0, 1).

§ 5. For the case of a finite and continuous symmetrical kernel, $\mathrm{K}(x, y)$, it was shown by Hilbert and Schmidt† that this equality (2) holds, provided the series is uniformly convergent with regard to both variables in the given interval.

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* Cf. Bôcher, *An Introduction to the Study of Integral Equations* (*Camb. Tracts, No.* 10), p. 57.

† Cf. Schmidt, *Zur Theorie der linearen und nichtlinearen Integralgleichungen, Math. Ann. Bd.* 63, p. 449, 1907.

Kneser has extended* the theorem to kernels which become infinite as $\frac{1}{r}$, and proved that the Green's Function, as defined in § 1, is given by the infinite series

$$\frac{\psi_1(0)\psi_1(1)}{\lambda_1} + \frac{\psi_2(0)\psi_2(1)}{\lambda_2} + \ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

provided that the series converges uniformly within S, both with respect to (x, y, z) and (x_1, y_1, z_1), when the distance r_{01} between (0) and (1) remains greater than some fixed quantity.

In some cases it is not difficult to prove the uniform convergence of the series, so that the Green's Functions are definitely established by means of this theorem. Various examples of this are given in Kneser's book.

In other cases, however, we have to be content with the position that the Integral Equation (1) suggests the expression for the Green's Function, which has then to be verified independently.

It should be noted that in these cases, *repeated* series, in which the summations are taken one after the other, are substituted for the original double, or triple, series.†

In the following pages we shall show that the series (2) does give the Green's Function in the Theory of Potential, when the region is bounded by concentric circular cylinders, planes perpendicular to the axis, and axial planes : and by concentric spheres, axial planes and cones. The Green's Functions for these cases have been obtained by other methods by Dougall.‡ They can also be deduced from results given by me§ for Instantaneous Point Sources in the Conduction of Heat, by replacing the *instantaneous* source by a constant and continuous one.

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* Kneser, *Die Theorie der Integralgleichungen und die Darstellung willkürlicher Funktionen in der mathematischen Physik, Math. Ann. Bd.* 63, p. 486, 1907. Also *Integralgleichungen*, § 41.

† Cf. Kneser, *Integralgleichungen und Darstellung willkürlicher Funktionen von zwei Variabeln, Rend. Circ. Mat. di Palermo*, T. XXVII., p. 117, 1909.

‡ Dougall, *The Determination of Green's Functions by means of Cylindrical and Spherical Harmonics, Proc. Edin. Math. Soc.*, Vol. XVIII., p. 33, 1900.

§ *The Green's Functions for a Wedge of any Angle and other Problems in the Conduction of Heat, Proc. Lond. Math. Soc.* (2), Vol. VIII., p. 365, 1910.

§ 6. The following theorems dealing with Bessel's Functions will be required later:—

I. *Let* $\rho_1, \rho_2, \ldots$ *be the positive roots of the equation*

$$J_n(\rho a) = 0. \qquad (n > 0).$$

Then

$$\sum_{\rho} \frac{J_n(\rho x) J_n(\rho \xi)}{\rho^2 \int_0^a x J_n^2(\rho x)\, dx} = \frac{1}{2n}\left(\frac{x}{a}\right)^n \left\{\left(\frac{a}{\xi}\right)^n - \left(\frac{\xi}{a}\right)^n\right\} \quad \textit{for } 0 < x < \xi$$

$$= \frac{1}{2n}\left(\frac{\xi}{a}\right)^n \left\{\left(\frac{a}{x}\right)^n - \left(\frac{x}{a}\right)^n\right\} \quad \textit{for } \xi < x < a.*$$

II. *Let* $\rho_1, \rho_2, \ldots$ *be the positive roots of the equation*

$$J_n(\rho a) K_n(i\rho b) - J_n(\rho b) K_n(i\rho a) = 0, \quad 0 < a < b, \dagger$$
$$n > 0;$$

and let $U_n(\rho x)$ *stand for the expression*

$$J_n(\rho x) K_n(i\rho b) - J_n(\rho b) K_n(i\rho x).$$

Then

$$\sum_{\rho} \frac{U_n(\rho x) U_n(\rho \xi)}{\rho^2 \int_a^b x U_n^2(\rho x)\, dx} = \frac{\frac{1}{2n}\left\{\left(\frac{x}{a}\right)^n - \left(\frac{a}{x}\right)^n\right\}\left\{\left(\frac{b}{\xi}\right)^n - \left(\frac{\xi}{b}\right)^n\right\}}{\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n}$$

for $a < x < \xi < b,$

$$= \frac{\frac{1}{2n}\left\{\left(\frac{\xi}{a}\right)^n - \left(\frac{a}{\xi}\right)^n\right\}\left\{\left(\frac{b}{x}\right)^n - \left(\frac{x}{b}\right)^n\right\}}{\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n}$$

for $a < \xi < x < b.$

The second of these theorems can be proved by an argument similar to that by which Kneser establishes the first. The value

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* Cf. Kneser, *Integralgleichungen*, § 27.

† As usual $K_n(ix)$ stands for $\frac{\pi}{2\sin n\pi} e^{-\frac{1}{2}ni\pi}\left(J_{-n}(x) - e^{ni\pi} J_n(x)\right)$.

of $U_n(x)$ for large values of x is needed, and the fact that, when ρ is large, the root ρ_s of the equation

$$J_n(\rho a)K_n(i\rho b) - J_n(\rho b)K_n(i\rho a) = 0$$

is approximately $\dfrac{sa\pi}{b-a}$.*

III. *If ρ, ρ' are two different positive roots of*

$$U_n(\rho a) = J_n(\rho a)K_n(i\rho b) - J_n(\rho b)K_n(i\rho a) = 0, \quad n > 0$$

(i) $$\int_a^b x U_n(\rho x) U_n(\rho' x)\,dx = 0$$

and (ii) $$\int_a^b x U_n^2(\rho x)\,dx = -\frac{a}{2\rho}\frac{d}{d\rho}U_n(\rho a)\frac{d}{da}U_n(\rho a)$$

The proof of this theorem offers no difficulty.†

Cylindrical Coordinates.

§ 7. We proceed to examine the different series

$$\frac{\psi_1(0)\psi_1(1)}{\lambda_1} + \frac{\psi_2(0)\psi_2(1)}{\lambda_2} + \dots$$

for the regions named in § 5.

We begin with those in which cylindrical coordinates (r, θ, z) are used. With these coordinates the equation

$$\nabla^2\psi + \lambda\psi = 0$$

becomes

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} + \lambda\psi = 0.$$

Space bounded by the cylinder $r = a$, and the planes $z = 0$, $z = c$.

The finite and continuous solutions vanishing at the boundary are given by

$$\psi_{\rho mn} = A_{\rho mn} J_m(\rho r) \begin{matrix}\cos \\ \sin\end{matrix} m\theta \sin\frac{n\pi}{c}z,$$

where m, n are positive integers, including zero, ρ is a positive root of $J_m(\rho a) = 0$, and $\lambda = \rho^2 + \dfrac{n^2\pi^2}{c^2}$.

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* Cf. Gray and Mathews, *Bessel Functions*, p. 242.

† Cf. Carslaw, *Fourier's Series*, p. 315.

To normalise these functions we choose $A_{\rho mn}$ so that

$$A^2_{\rho mn}\int_0^a rJ^2_m(\rho r)dr\int_0^{2\pi}\begin{matrix}\cos^2\\ \sin^2\end{matrix}m\theta d\theta\int_0^c \sin^2\frac{n\pi}{c}zdz=1.$$

Thus $$A^2_{\rho mn}=\frac{a_m}{\pi c\int_0^a rJ^2_m(\rho r)\,dr},$$

where $a_0=1$ and $a_m=2$, for $m\geqq 1$.

Therefore the series corresponding to

$$\Sigma\frac{\psi_s(0)\,\psi_s(1)}{\lambda_s}$$

is

$$\frac{1}{\pi c}\sum_\rho\sum_m\sum_n\frac{a_mJ_m(\rho r)\,J_m(\rho r')}{\left(\rho^2+\frac{n^2\pi^2}{c^2}\right)\int_0^a rJ^2_m(\rho r)dr}\cos m(\theta-\theta')\sin\frac{n\pi}{c}z\sin\frac{n\pi}{c}z'.$$

Now it is easy to show that

$$\left.\begin{aligned}\frac{2}{c}\sum_1^\infty\frac{\sin\frac{n\pi}{c}z\sin\frac{n\pi}{c}z'}{\rho^2+\frac{n^2\pi^2}{c^2}}&=\frac{1}{\rho}\frac{\sinh\rho(c-z)\sinh\rho z'}{\sinh\rho c},\quad\text{for}\quad 0<z'<z:\\ &=\frac{1}{\rho}\frac{\sinh\rho(c-z')\sinh\rho z}{\sinh\rho c},\quad\text{for}\quad z<z'<c\end{aligned}\right\}\ (4).$$

Therefore if we start with a repeated series, and sum first with regard to n, we have

$$\sum_m a_m\cos m(\theta-\theta')\sum_\rho\frac{J_m(\rho r)J_m(\rho r')}{2\pi\rho\int_0^a rJ^2_m(\rho r)dr}\frac{\sinh\rho(c-z)\sinh\rho z'}{\sinh\rho c}\quad\ldots\ldots\ldots\ldots(5)$$

for $z>z'$, and when $z<z'$, we have to interchange z and z' in (5).

But it is known* that

$$\int_0^a rJ^2_m(\rho r)dr=\frac{a^2}{2}[J'_m(\rho a)]^2,$$

when ρ is a root of $J_m(\rho a)=0$.

Therefore we can replace (5) by

$$\frac{1}{\pi a^2}\sum_m a_m\cos m(\theta-\theta')\sum_\rho\frac{J_m(\rho r)\,J_m(\rho r')}{\rho[J'_m(\rho a)]^2}\frac{\sinh\rho(c-z)\sinh\rho z'}{\sinh\rho c}\ \text{for}\ z>z'.\ (6)$$

This result agrees with Dougall, *loc. cit.* § 22 (2).†

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* Cf. *Fourier's Series*, p. 315.

† In comparing Dougall's results with those given in this paper, it has to be remembered that his Green's Functions are infinite as $\frac{1}{r}$, not as $\frac{1}{4\pi r}$.

Corollary. We can deduce from (6) the Green's Functions for

(i) *The space bounded by a cylinder* $r = a$:

(ii) *The space between the planes* $z = 0$ *and* $z = c$.

In (6) change the origin to the point on the axis half-way between the planes.

Then we have

$$K(0, 1) = \frac{1}{\pi a^2} \sum_m a_m \cos m(\theta - \theta') \sum_\rho \frac{J_m(\rho r)\, J_m(\rho r')}{\rho [J'_m(\rho a)]^2} \frac{\sinh\rho\left(\frac{c}{2} - z\right)\sinh\rho\left(\frac{c}{2} + z'\right)}{\sinh\rho c} \quad \text{for } -\frac{c}{2} < z' < z < \frac{c}{2}.$$

Put $c = \infty$, and we obtain *the Green's Function for the cylinder*,

$$K(0, 1) = \frac{1}{2\pi a^2} \sum_m a_m \cos m(\theta - \theta') \sum_\rho \frac{J_m(\rho r)\, J_m(\rho r')}{\rho [J'_m(\rho a)]^2} e^{-\rho(z - z')} \quad \text{for } z > z' \ldots (7)$$

For the space between the two parallel planes, we put $a = \infty$ in (6).

Using the asymptotic value* of $J_m(\rho a)$, namely,

$$\frac{2}{\sqrt{\pi \rho a}} \cos\left(\frac{m\pi}{2} + \frac{\pi}{4} - \rho a\right),$$

we have

$$J'_m(\rho a) = \sqrt{\frac{2}{\pi \rho a}} \sin\left(\frac{m\pi}{2} + \frac{\pi}{4} - \rho a\right).$$

If $\rho, \rho + \delta\rho$ are consecutive roots of

$$J_m(\rho a) = 0,$$

we have

$$a\delta\rho = \pi$$

$$a^2 [J'_m(\rho a)]^2 = \frac{2a}{\pi \rho} = \frac{2}{\rho \delta\rho}.$$

Thus from (6)

$$K(0, 1) = \frac{1}{2\pi} \sum_m a_m \cos m(\theta - \theta') \int_0^\infty J_m(\rho r)\, J_m(\rho r') \frac{\sinh\rho(c - z)\sinh\rho z'}{\sinh\rho c} d\rho,$$

$$= \frac{1}{2\pi} \int_0^\infty \frac{\sinh\rho(c - z)\sinh\rho z'}{\sinh\rho c} J_0(\rho R) d\rho, \quad \text{for } z > z'. \ldots\ldots\ldots\ldots (8)$$

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* Cf. Macdonald, *The Electrical Distribution on a Conductor bounded by Two Spherical Surfaces cutting at any Angle, Proc. Lond. Math. Soc.*, (1) Vol. XXVI., p. 159, 1895.

§ 8. *Space bounded by a cylinder* $r=a$; *two planes* $z=0$, $z=c$; *and two axial planes* $\theta=0$, $\theta=\alpha$.

Here we have $\psi_{\rho mn}=\dfrac{2}{\sqrt{\alpha c}}\mathrm{J}_{\frac{m\pi}{\alpha}}(\beta r)\dfrac{\sin\frac{m\pi}{\alpha}\theta\sin\frac{n\pi}{c}z}{\left[\int_0^a r\mathrm{J}^2_{\frac{m\pi}{\alpha}}(\rho r)dr\right]^{\frac{1}{2}}}$,

the function having been normalised.

In this expression, m and n are positive integers; ρ is a positive root of $\mathrm{J}_{\frac{m\pi}{\alpha}}(\rho a)=0$; and $\lambda=\rho^2+\dfrac{n^2\pi^2}{c^2}$.

Therefore the series corresponding to

$$\Sigma\frac{\psi_s(0)\psi_s(1)}{\lambda_s}$$

is

$$\frac{4}{\alpha c}\Sigma_\rho\Sigma_m\Sigma_n\frac{\mathrm{J}_{\frac{m\pi}{\alpha}}(\rho r)\mathrm{J}_{\frac{m\pi}{\alpha}}(\rho r')}{\left(\rho^2+\frac{n^2\pi^2}{c^2}\right)\int_0^a r\mathrm{J}^2_{\frac{m\pi}{\alpha}}(\rho r)dr}\sin\frac{n\pi}{\alpha}\theta\sin\frac{m\pi}{\alpha}\theta'\sin\frac{n\pi}{c}z\sin\frac{n\pi}{c}z'.$$

If we sum first with regard to z, as in § 7, we have

$$\frac{4}{\alpha a^2}\Sigma_m\sin\frac{m\pi}{\alpha}\theta\sin\frac{m\pi}{\alpha}\theta'\Sigma_\rho\frac{\mathrm{J}_{\frac{m\pi}{\alpha}}(\rho r)\mathrm{J}_{\frac{m\pi}{\alpha}}(\rho r')}{\rho\left[\mathrm{J}'_{\frac{m\pi}{\alpha}}(\rho a)\right]^2}\frac{\sinh\rho(c-z)\sinh\rho z'}{\sinh\rho c},$$

for $z>z'$. (9)

This agrees with Dougall, *loc. cit.* § 25 (1).

Corollary. We can deduce from (9) the Green's Functions for

(i) *The space bounded by a cylinder and two axial planes*:

(ii) *The wedge of angle* α:

(iii) *The wedge of angle* α *and two planes* $z=0$, $z=c$

For these we have, respectively,

$$\mathrm{K}(0,\,1)=\frac{2}{\alpha a^2}\Sigma_m\sin\frac{m\pi}{\alpha}\theta\sin\frac{m\pi}{\alpha}\theta'\sum_\rho\frac{\mathrm{J}_{\frac{m\pi}{\alpha}}(\rho r)\mathrm{J}_{\frac{m\pi}{\alpha}}(\rho r')}{\rho\int_0^a r\mathrm{J}^2_{\frac{m\pi}{\alpha}}(\rho r)dr}e^{-\rho(z-z')}$$

for $z>z'$: (10)

$$K(0,1)=\frac{1}{\alpha}\sum_m \sin\frac{m\pi}{\alpha}\theta\sin\frac{m\pi}{\alpha}\theta'\int_0^\infty J_{\frac{m\pi}{\alpha}}(\rho r)J_{\frac{m\pi}{\alpha}}(\rho r')\,e^{-\rho(z-z')}\,d\rho$$

for $z>z'$: (11)

$$K(0,1)=\frac{2}{\alpha}\sum_m \sin\frac{m\pi}{\alpha}\theta\sin\frac{m\pi}{\alpha}\theta'\int_0^\infty J_{\frac{m\pi}{\alpha}}(\rho r)J_{\frac{m\pi}{\alpha}}(\rho r')\,\frac{\sinh\rho(c-z)\sinh\rho z'}{\sinh\rho c}\,d\rho \quad \text{for } z>z'. \quad (12)$$

§ 9. *Space bounded by two cylinders* $r=a$, $r=b$: *and two planes* $z=0$, $z=c$.

Here the normalised function

$$\psi_{\rho mn}=\sqrt{\frac{a_m}{\pi c}}\frac{U_m(\rho r)\overset{\cos}{\sin}m\theta\,\sin\frac{n\pi}{c}z}{\left[\int_a^b rU_m^2(\rho r)dr\right]^{\frac{1}{2}}},$$

where m, n are positive integers, or zero:

$U_m(\rho r)=J_m(\rho r)K_m(i\rho b)-J_m(\rho b)K_m(i\rho r)$:

ρ is a positive root of $U_m(\rho a)=0$:

$a_0=1$, $a_m=2$ for $m\geqq 1$:

and $\lambda=\rho^2+\frac{n^2\pi^2}{c^2}$.

Thus the series corresponding to

$$\sum\frac{\psi_s(0)\,\psi_s(1)}{\lambda_s}$$

is

$$\frac{1}{\pi c}\sum_\rho\sum_m\sum_n\frac{a_mU_m(\rho r)U_m(\rho r')}{\left(\rho^2+\frac{n^2\pi^2}{c^2}\right)\int_a^b rU_m^2(\rho r)dr}\cos m(\theta-\theta')\sin\frac{n\pi}{c}z\,\sin\frac{n\pi}{c}z'.$$

If we sum first with regard to n, we have

$$\frac{1}{2\pi}\sum_m a_m\cos m(\theta-\theta')\sum_\rho\frac{U_m(\rho r)U_m(\rho r')}{\rho\int_a^b rU_m^2(\rho r)dr}\,\frac{\sinh\rho(c-z)\sinh\rho z'}{\sinh\rho c}\quad\text{for } z>z'. \quad (13)$$

Using the result of § 6 (III.), we have

$$-\frac{1}{\pi}\sum_m a_m \cos m(\theta-\theta')\sum_\rho \frac{U_m(\rho r)U_m(\rho r')}{\frac{d}{d\rho}U_m(\rho a)\frac{d}{da}U_n(\rho a)}\cdot\frac{\sinh\rho(c-z)\sinh\rho z'}{\sinh\rho c}$$

for $z>z'$. (14)

Corollary. For the space bounded by two cylinders we find from (14)

$$K(0,1)=-\frac{1}{2\pi}\Sigma a_m \cos m(\theta-\theta')\sum_\rho \frac{U_m(\rho r)U_m(\rho r')}{\frac{d}{d\rho}U_m(\rho a)\frac{d}{da}U_m(\rho a)}e^{-\rho(z-z')}$$

for $z>z'$. (15)

§ 10. *Space bounded by two cylinders* $r=a$, $r=b$: *two planes* $z=0$, $z=c$: *and two axial planes* $\theta=0$, $\theta=\alpha$.

Here the normalised function

$$\psi_{\rho mn}=\frac{2}{\sqrt{\alpha c}}\cdot\frac{U_{\frac{m\pi}{\alpha}}(\rho r)\sin\frac{m\pi}{\alpha}\theta\sin\frac{n\pi}{c}z}{\left[\int_a^b rU^2_{\frac{m\pi}{\alpha}}(\rho r)dr\right]^{\frac{1}{2}}},$$

where $U_{\frac{m\pi}{\alpha}}(\rho r)=J_{\frac{m\pi}{\alpha}}(\rho r)K_{\frac{m\pi}{\alpha}}(i\rho b)-J_{\frac{m\pi}{\alpha}}(\rho b)K_{\frac{m\pi}{\alpha}}(i\rho r)$:

m, n are positive integers:

ρ is a positive root of $U_{\frac{m\pi}{\alpha}}(\rho a)=0$:

and $\lambda=\rho^2+\frac{n^2\pi^2}{c^2}$.

Thus the series corresponding to

$$\Sigma\frac{\psi_s(0)\,\psi_s(1)}{\lambda_s}$$

is

$$\frac{4}{\alpha c}\Sigma\Sigma\Sigma_{\rho\,m\,n}\frac{U_{\frac{m\pi}{\alpha}}(\rho r)U_{\frac{m\pi}{\alpha}}(\rho r')}{\left(\rho^2+\frac{n^2\pi^2}{c^2}\right)\int_a^b rU^2_{\frac{m\pi}{\alpha}}(\rho r)dr}\sin\frac{m\pi}{\alpha}\theta\sin\frac{m\pi}{\alpha}\theta'\sin\frac{n\pi}{c}z\sin\frac{n\pi}{c}z'.$$

If we sum first with regard to n, we have

$$-\frac{4}{a\alpha}\sum_m \sin\frac{m\pi}{\alpha}\theta\sin\frac{m\pi}{\alpha}\theta'\sum_\rho \frac{\mathrm{U}_{\frac{m\pi}{\alpha}}(\rho r)\mathrm{U}_{\frac{m\pi}{\alpha}}(\rho r')}{\frac{d}{d\rho}\mathrm{U}_{\frac{m\pi}{\alpha}}(\rho a)\frac{d}{da}\mathrm{U}_{\frac{m\pi}{\alpha}}(\rho a)}\cdot\frac{\sinh\rho(c-z)\sinh\rho z'}{\sinh\rho c},$$

for $z>z'$. (16)

Dougall, *loc. cit.* § 26(2) obtains a different result.

Spherical Coordinates.

§ 11. Before dealing with the regions bounded by spheres, axial planes and cones, it will be necessary to state some theorems regarding the Spherical Harmonics we employ.

Following Hobson* we define $\mathrm{P}_n^{-m}(\mu)$ by the equation

$$\mathrm{P}_n^{-m}(\mu)=\frac{1}{\Pi(m)}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{m}{2}}\mathrm{F}\left(n+1,\ -n,\ m+1,\ \frac{1-\mu}{2}\right).$$

This solution of the equation

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{d\mathrm{P}}{d\mu}\right)+\left(n(n+1)-\frac{m^2}{1-\mu^2}\right)\mathrm{P}=0$$

is finite at $\mu=1(\theta=0)$, and is suitable for the potential problems to be discussed.

I. *When m is any positive number, and n, n′ are roots of*

$$\mathrm{P}_n^{-m}(\mu_0)=0,$$

then (i)

$$\int_{\mu_0}^{1}\mathrm{P}_n^{-m}(\mu)\mathrm{P}_{n'}^{-m}(\mu)d\mu=0,$$

and (ii)

$$\int_{\mu_0}^{1}[\mathrm{P}_n^{-m}(\mu)]^2d\mu=-\frac{(1-\mu_0^2)}{2n+1}\frac{d}{dn}\cdot\mathrm{P}_n^{-m}(\mu_0)\frac{d}{d\mu_0}\cdot\mathrm{P}_n^{-m}(\mu_0).$$

The proof of this theorem resembles that given in Macdonald's *Electric Waves*, p. 89, for a special case.

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* Hobson, *On a Type of Spherical Harmonics of Unrestricted Degree, Order, and Argument*, Phil. Trans. (A), Vol. 187, p. 451, 1896.

II. *If m is any positive number, and $\pi - \theta = \epsilon$ where ϵ is small, the positive roots of*

$$P_n^{-m}(\mu) = 0$$

are given by

$$n = m + s + \frac{\Pi(2m+s)}{\Pi(m)\Pi(m-1)\Pi(s)} \tan^{2m}\frac{\epsilon}{2},$$

where s has all positive integral values, including zero.

In the limit when $\epsilon \to 0$, $n = m + s$.

This theorem is due to Macdonald.*

III. *If n is any positive root of $P_n^{-m}(\mu) = 0$ and m is any positive number,*

$$(1-\mu^2)\frac{d}{dn}P_n^{-m}(\mu) = \frac{2}{\pi P_n^{-m}(-\mu)} \frac{\Pi(n-m)}{\Pi(n+m)} \sin(n-m)\pi.$$

This theorem is proved in Dougall's paper, § 11, but his proof needs modification for the case of m zero or a positive integer.

IV. *If n is any positive root of $P_n^{-m}(\mu) = 0$ and m is any positive number, or zero,*

$$\underset{\mu \to -1}{\text{Lt.}} \left[(1-\mu^2)\frac{d}{dn}\left(P_n^{-m}(\mu)\frac{d}{d\mu}P_n^{-m}(\mu)\right)\right] = -2\frac{\Pi(n-m)}{\Pi(n+m)}.$$

Starting from the equation

$$P_n^{-m}(\mu) = \frac{\sin n\pi}{\sin m\pi} P_n^{-m}(\mu') - \frac{\Pi(n-m)}{\Pi(n+m)} \frac{\sin(n-m)\pi}{\sin m\pi} P_n^m(\mu'),$$

where $\mu = -\mu'$, [cf. Macdonald, *Proc. Lond. Math. Soc.*, (1) Vol. XXXI., p. 273] this theorem follows from an argument similar to that used by Macdonald, [cf. *Electric Waves*, § 63]. The result in (III.) is required in this proof.

V. *If n, n' are any two positive roots of the equation*

$$P_n^{-m}(\mu_1)P_n^m(\mu_0) - P_n^m(\mu_1)P_n^{-m}(\mu_0) = 0, \; m > 0$$

and

$$S_n^m(\mu) = P_n^{-m}(\mu)P_n^m(\mu_0) - P_n^m(\mu)P_n^{-m}(\mu_0),$$

then

$$\int_{\mu_0}^{\mu_1} S_n^m(\mu) S_{n'}^m(\mu) d\mu = 0$$

and

$$(2n+1)\int_{\mu_0}^{\mu_1} [S_n^m(\mu)]^2 d\mu = \left[(1-\mu^2)\frac{d}{dn}S_n^m(\mu)\frac{d}{d\mu}S_n^m(\mu)\right]_{\mu_0}^{\mu_1}$$

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* Macdonald, *Zeroes of the Spherical Harmonic $P_n^m(\mu)$ considered as a function of n*, *Proc. Lond. Math. Soc.* (1), Vol. XXXI., p. 273, 1899.

§ 12. *Space bounded by the sphere* $r = a$.

In Spherical Coordinates (r, θ, ϕ), the equation

$$\nabla^2\psi + \lambda\psi = 0$$

becomes

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\psi}{\partial\mu}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \lambda\psi = 0.$$

Thus we have

$$\psi_{\rho mn} = \mathrm{A}_{\rho mn}\frac{\mathrm{J}_{n+\frac{1}{2}}(\rho r)}{\sqrt{r}}\mathrm{P}_n^m(\mu)\begin{matrix}\cos\\ \sin\end{matrix} m\phi,$$

when m is any positive integer, or zero :

n is any positive integer :

ρ is any positive root of $\mathrm{J}_{n+\frac{1}{2}}(\rho a) = 0$:

and $\lambda = \rho^2$.

The function having been normalised, we find that the series corresponding to

$$\frac{\psi_1(0)\psi_1(1)}{\lambda_1} + \frac{\psi_2(0)\psi_2(1)}{\lambda_2} + \ldots$$

is

$$\frac{1}{2\pi}\sum_\rho\sum_n \frac{(rr')^{-\frac{1}{2}}(n+\frac{1}{2})\,\mathrm{J}_{n+\frac{1}{2}}(\rho r)\,\mathrm{J}_{n+\frac{1}{2}}(\rho r')\,\mathrm{P}_n(\cos\gamma)}{\rho^2\int_0^a r\mathrm{J}^2_{n+\frac{1}{2}}(\rho r)\,dr} \quad \ldots\ldots\ldots\ldots\ldots (17)$$

Kneser* examines this case in detail, and verifies that the repeated series

$$\frac{1}{2\pi\sqrt{rr'}}\sum_n (n+\tfrac{1}{2})\mathrm{P}_n(\cos\gamma)\sum_\rho \frac{\mathrm{J}_{n+\frac{1}{2}}(\rho r)}{\rho^2\int_0^a r\mathrm{J}^2_{n+\frac{1}{2}}(\rho r)dr}\frac{\mathrm{J}_{n+\frac{1}{2}}(\rho r')}{}$$

is equal to the known Green's Function.

§ 13. *Space bounded by the sphere* $r = a$ *and the cone* $\theta = \theta_0$.

Here the normalised function

$$\psi_{\rho mn} = \sqrt{\frac{a_m}{2\pi r}}\frac{\mathrm{J}_{n+\frac{1}{2}}(\rho r)\,\mathrm{P}_n^{-m}(\mu)\begin{matrix}\cos\\ \sin\end{matrix} m\phi}{\left[\int_0^a r\mathrm{J}^2_{n+\frac{1}{2}}(\rho r)dr\int_{\mu_0}^1 [\mathrm{P}_n^{-m}(\mu)]^2 d\mu\right]^{\frac{1}{2}}};$$

---

* Kneser, *Integralgleichungen*, § 38.

m being any positive integer, or zero :
n any positive root of $P_n^{-m}(\mu_0) = 0$:
ρ any positive root of $J_{n+\frac{1}{2}}(\rho a) = 0$:
$\lambda = \rho^2$, and $a_0 = 1$, $a_m = 2$ for $m \geq 1$.

Thus the series corresponding to

$$\frac{\psi_1(0)\,\psi_1(1)}{\lambda_1} + \frac{\psi_2(0)\,\psi_2(1)}{\lambda_2} + \dots$$

is

$$\sum_{\rho}\sum_{m}\sum_{n} \frac{a_m}{2\pi\sqrt{rr'}} \frac{J_{n+\frac{1}{2}}(\rho r)\,J_{n+\frac{1}{2}}(\rho r')}{\rho^2 \int_0^a r J_{n+\frac{1}{2}}^2(\rho r)\,dr} \frac{P_n^{-m}(\mu)\,P_n^{-m}(\mu')}{\int_{\mu_0}^1 [P_n^{-m}(\mu)]^2 d\mu} \cos m(\phi - \phi').$$

If we sum first with regard to ρ, and use the results of § 6 (I) and § 11 (I), we obtain

$$\frac{1}{2\pi\sqrt{rr'}\,(1-\mu_0^2)} \sum_m a_m \cos m(\phi - \phi') \sum_n \left(\frac{r}{a}\right)^{n+\frac{1}{2}} \frac{\left(\frac{a}{r'}\right)^{n+\frac{1}{2}} - \left(\frac{r'}{a}\right)^{n+\frac{1}{2}}}{\frac{d}{dn}P_n^{-m}(\mu_0)\frac{d}{d\mu_0}P_n^{-m}(\mu_0)} P_n^{-m}(\mu) P_n^{-m}(\mu').$$

for $0 < r < r' < a$.* (18)

Collorary. Put $a = \infty$ in (18) and we find that in the cone $\theta = \theta_0$

$$K(0, 1) = -\frac{1}{2\pi\sqrt{rr'}(1-\mu_0^2)} \sum_m a_m \cos m(\phi - \phi') \sum_n \left(\frac{r}{r'}\right)^{n+\frac{1}{2}} \frac{P_n^{-m}(\mu)\,P_n^{-m}(\mu')}{\frac{d}{dn}P_n^{-m}(\mu_0)\frac{d}{d\mu_0}P_n^{-m}(\mu_0)} \text{ for } r < r$$

(19)

This agrees with Dougall, *loc. cit.* § 29 (2).

§ 14. *Space bounded by the sphere $r = a$, the cone $\theta = \theta_0$, and the planes $\phi = 0$, $\phi = \alpha$.*

Here $\psi_{\rho mn} = A_{\rho mn} \frac{J_{n+\frac{1}{2}}(\rho r)}{\sqrt{r}} P_n^{-\frac{m\pi}{\alpha}}(\mu) \sin\frac{m\pi}{\alpha}\phi,$

where m is any positive integer :

n is any positive root of $P_n^{-\frac{m\pi}{\alpha}}(\mu_0) = 0$:
ρ is any positive root of $J_{n+\frac{1}{2}}(\rho a) = 0$:
and $\lambda = \rho^2$.

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* Cf. Macdonald, *Demonstration of Green's Formula for Electric Density near the Vertex of a Right Cone, Trans. Camb. Phil. Soc., Vol. XVIII., p.* 293, 1900. The Green's Function for $\theta' = 0$ is found in that paper.

We are thus led to the series

$$\frac{2}{\alpha\sqrt{rr'}}\sum\sum\sum_{\rho\,m\,n}\frac{J_{n+\frac{1}{2}}(\rho r)J_{n+\frac{1}{2}}(\rho r')}{\rho^2\int_0^a rJ^2_{n+\frac{1}{2}}(\rho r)dr}\frac{P_n^{-\frac{m\pi}{\alpha}}(\mu)P_n^{-\frac{m\pi}{\alpha}}(\mu')}{\int_{\mu_0}^1[P_n^{-\frac{m\pi}{\alpha}}(\mu)]^2d\mu}\sin\frac{m\pi}{\alpha}\phi\sin\frac{m\pi}{\alpha}\phi'.$$

If we sum first with regard to ρ, using § 6 (I.) and § 11 (I.) we have

$$-\frac{2}{\alpha\sqrt{rr'(1-\mu_0^2)}}\sum_m\sin\frac{m\pi}{\alpha}\phi\sin\frac{m\pi}{\alpha}\phi'\sum_n\left(\frac{r}{a}\right)^{n+\frac{1}{2}}\left[\left(\frac{a}{r'}\right)^{n+\frac{1}{2}}-\left(\frac{r'}{a}\right)^{n+\frac{1}{2}}\right]\frac{P_n^{-m}(\mu)P_n^{-m}(\mu')}{\frac{d}{dn}P_n^{-m}(\mu_0)\frac{d}{d\mu_0}P_n^{-m}(\mu_0)}$$

for $0<r<r'<a$. (20)

Corollary (i.) Put $a=\infty$ in (20) and we find for *the space bounded by the cone* $\theta=\theta_0$ *and the two planes* $\phi=0$ *and* $\phi=\alpha$,

$$K(0,1)=-\frac{2}{\alpha\sqrt{rr'(1-\mu_0^2)}}\sum_m\sin\frac{m\pi}{\alpha}\phi\sin\frac{m\pi}{\alpha}\phi'\sum_n\left(\frac{r}{r'}\right)^{n+\frac{1}{2}}\frac{P_n^{-\frac{m\pi}{\alpha}}(\mu)P_n^{-\frac{m\pi}{\alpha}}(\mu')}{\frac{d}{dn}P_n^{-\frac{m\pi}{\alpha}}(\mu_0)\frac{d}{d\mu_0}P_n^{-\frac{m\pi}{\alpha}}(\mu_0)}$$

for $r<r'$. (21)

There is an obvious error in Dougall's result, *loc. cit.* § 33 (2), $S_n^m(\beta)$ being omitted.

Corollary (ii.) Let $\theta_0\to\pi$ and we find from (21), with the aid of § 11 (II.), (IV.) *for the space bounded by the planes* $\phi=0$, $\phi=\alpha$,

$$K(0,1)=\frac{1}{\alpha\sqrt{rr'}}\sum_m\sin\frac{m\pi}{\alpha}\phi\sin\frac{m\pi}{\alpha}\phi'\sum_s\frac{\Pi\left(s+2\frac{m\pi}{\alpha}\right)}{\Pi(s)}\left(\frac{r}{r'}\right)^{\frac{m\pi}{\alpha}+s+\frac{1}{2}}P^{-\frac{m\pi}{\alpha}}_{\frac{m\pi}{\alpha}+s}(\mu)\,P^{-\frac{m\pi}{\alpha}}_{\frac{m\pi}{\alpha}+s}(\mu'$$

for $r<r'$, (2

s being any positive integer, including zero.

This agrees with Dougall, *loc. cit.* § 30 (3).

§ 14. *Space bounded by two spheres* $r=a$, $r=b$.

Here $\psi_{\rho mn}=A_{\rho mn}\frac{J_{n+\frac{1}{2}}(\rho r)K_{n+\frac{1}{2}}(i\rho b)-J_{n+\frac{1}{2}}(\rho b)K_{n+\frac{1}{2}}(i\rho r)}{\sqrt{r}}P_n^m(\mu)\begin{matrix}\cos\\ \sin\end{matrix}m\phi,$

where m is any positive integer, including zero:

n is any positive integer:

ρ is any positive root of $J_{n+\frac{1}{2}}(\rho a)K_{n+\frac{1}{2}}(i\rho b) - J_{n+\frac{1}{2}}(\rho b)K_{n+\frac{1}{2}}(i\rho a) = 0$:

and $\lambda = \rho^2$.

We are thus led to the series

$$\frac{1}{4\pi}\sum_{\rho}\sum_{n}\frac{U_{n+\frac{1}{2}}(\rho r)U_{n+\frac{1}{2}}(\rho r')}{\rho^2\int_0^a r U^2_{n+\frac{1}{2}}(\rho r)dr}(2n+1)P_n(\cos\gamma),$$

writing, as in § 6, $U_n(\rho x)$ for $J_n(\rho x)K_n(i\rho b) - J_n(\rho b)K_n(i\rho x)$.

If we sum first with regard to ρ, using § 6 (II.) and put $r = ae^{\eta}$, $r' = ae^{\eta'}$, and $b = ae^{c}$, we obtain

$$\frac{1}{2\pi\sqrt{rr'}}\sum_{n}\sinh(n+\tfrac{1}{2})\eta\frac{\sinh(n+\frac{1}{2})(c-\eta')}{\sinh(n+\frac{1}{2})c}P_n(\cos\gamma) \text{ for } \eta<\eta'. \qquad (23)$$

This agrees with Dougall, *loc. cit.* § 28(1).

§ 15. *Space bounded by two spheres $r = a$, $r = b$, and the cone $\theta = \theta_0$.*

$$\text{Here } \psi_{\rho mn} = A_{\rho mn}\frac{U_{n+\frac{1}{2}}(\rho r)}{\sqrt{r}}P_n^{-\frac{m\pi}{\alpha}}(\mu)\begin{matrix}\cos\\ \sin\end{matrix} m\phi,$$

where m is any positive integer, including zero:

n is any positive root of $P_n^{-\frac{m\pi}{\alpha}}(\mu_0) = 0$:

ρ is any positive root of $U_{n+\frac{1}{2}}(\rho a) = 0$:

and $\lambda = \rho^2$.

On normalising the function, and proceeding as in § 14, we obtain

$$-\frac{1}{\pi\sqrt{rr'}}\sum_{m}a_m\cos m(\phi-\phi')\sum_{n}\frac{\sinh(n+\frac{1}{2})\eta\sinh(n+\frac{1}{2})(c-\eta')}{\sinh(n+\frac{1}{2})c}$$

$$\frac{P_n^{-\frac{m\pi}{\alpha}}(\mu)P_n^{-\frac{m\pi}{\alpha}}(\mu')}{(1-\mu_0^2)\frac{d}{dn}P_n^{-\frac{m\pi}{\alpha}}(\mu_0)\frac{d}{d\mu_0}P_n^{-\frac{m\pi}{\alpha}}(\mu_0)}, \qquad (24)$$

where $a_0 = 1$, $a_m = 2$ for $m \geqq 1$ and $\eta < \eta'$.

This is equivalent to Dougall's result, *loc. cit.* § 31 (2).

§16. *Space bounded by two spheres, $r=a$, $r=b$; two axial planes $\phi=0$, $\phi=\alpha$; and the cone $\theta=\theta_0$.*

Here we are led to the series

$$\frac{2}{\alpha\sqrt{rr'}}\sum_{\rho}\sum_{m}\sum_{n}\frac{U_{n+\frac{1}{2}}(\rho r)U_{n+\frac{1}{2}}(\rho r')}{\rho^2\int_a^b rU^2_{n+\frac{1}{2}}(\rho r)dr}\,\frac{P_n^{-\frac{m\pi}{\alpha}}(\mu)P_n^{-\frac{m\pi}{\alpha}}(\mu')}{\int_{\mu_0}^{1}[P_n^{-\frac{m\pi}{\alpha}}(\mu)]^2d\mu}\sin\frac{m\pi}{\alpha}\phi\sin\frac{m\pi}{\alpha}\phi'.$$

Where m is any positive integer;

n is any positive root of $P_n^{-\frac{m\pi}{\alpha}}(\mu_0)=0$:

ρ is any positive root of $U_{n+\frac{1}{2}}(\rho a)=0$:

and $\lambda=\rho^2$.

Proceeding as in §15 we obtain, for $\eta<\eta'$,

$$-\frac{4}{\alpha\sqrt{rr'}}\sum_{m}\sin\frac{m\pi}{\alpha}\phi\sin\frac{m\pi}{\alpha}\phi'\sum_{n}\frac{\sinh(n+\frac{1}{2})\eta\sinh(n+\frac{1}{2})(c-\eta')}{\sinh(n+\frac{1}{2})c}$$

$$\frac{P_n^{-\frac{m\pi}{\alpha}}(\mu)P_n^{-\frac{m\pi}{\alpha}}(\mu')}{(1-\mu_0^2)\frac{d}{dn}P_n^{-\frac{m\pi}{\alpha}}(\mu_0)\frac{d}{d\mu_0}P_n^{-\frac{m\pi}{\alpha}}(\mu_0)}, \quad (25).$$

Corollary. Let $\theta_0\to\pi$, and from §11 (II.) and (IV.) we obtain for *the space between two spheres and two axial planes*

$$K(0,1)=\frac{2}{\alpha\sqrt{rr'}}\sum_{m}\sin\frac{m\pi}{\alpha}\phi\sin\frac{m\pi}{\alpha}\phi'\sum_{s}\frac{\sinh\left(\frac{m\pi}{\alpha}+s+\frac{1}{2}\right)\eta\sinh\left(\frac{m\pi}{\alpha}+s+\frac{1}{2}\right)(c-\eta')}{\sinh\left(\frac{m\pi}{\alpha}+s+\frac{1}{2}\right)c}$$

$$\frac{\Pi\left(2\frac{m\pi}{\alpha}+s\right)}{\Pi(s)}P^{-\frac{m\pi}{\alpha}}_{\frac{m\pi}{\alpha}+s}(\mu)P^{-\frac{m\pi}{\alpha}}_{\frac{m\pi}{\alpha}+s}(\mu'), \quad (26$$

s being any positive integer, including zero, and $\eta<\eta'$.

§17. *Space bounded by two spheres $r=a$, $r=b$: two cones $\theta=\theta_0$, $\theta=\theta_1$: and two planes $\phi=0$, $\phi=\alpha$.*

Here we are led to the series

$$\frac{2}{\alpha\sqrt{rr'}}\sum_{\rho}\sum_{m}\sum_{n}\frac{\mathrm{U}_{n+\frac{1}{2}}(\rho r)\,\mathrm{U}_{n+\frac{1}{2}}(\rho r')}{\rho^2\int_a^b r\mathrm{U}^2_{n+\frac{1}{2}}(\rho r)dr}\cdot\frac{\mathrm{S}_n^{\frac{m\pi}{\alpha}}(\mu)\,\mathrm{S}_n^{\frac{m\pi}{\alpha}}(\mu')}{\int_{\mu_1}^{\mu_0}\left[\mathrm{S}_n^{\frac{m\pi}{\alpha}}(\mu)\right]^2 d\mu}\sin\frac{m\pi}{\alpha}\phi\sin\frac{m\pi}{\alpha}\phi'.$$

where

m is any positive integer :

n is any positive root of $\mathrm{P}_n^{-\frac{m\pi}{\alpha}}(\mu_1)\mathrm{P}_n^{\frac{m\pi}{\alpha}}(\mu_0)-\mathrm{P}_n^{-\frac{m\pi}{\alpha}}(\mu_0)\mathrm{P}_n^{\frac{m\pi}{\alpha}}(\mu_1)=0$,

ρ is any positive root of $\mathrm{U}_{n+\frac{1}{2}}(\rho a)=0$:

$\mathrm{S}_n^m(\mu)$ denotes $\mathrm{P}_n^{-\frac{m\pi}{\alpha}}(\mu)\mathrm{P}_n^{\frac{m\pi}{\alpha}}(\mu_0)-\mathrm{P}_n^{\frac{m\pi}{\alpha}}(\mu)\mathrm{P}_n^{-\frac{m\pi}{\alpha}}(\mu_0)$:

and $\lambda=\rho^2$.

Using § 6 (II.) and § 11 (V.), and summing first with regard to ρ, we obtain

$$\frac{4}{(1-\mu_0^2)\alpha\sqrt{rr'}}\sum_{m}\sin\frac{m\pi}{\alpha}\phi\sin\frac{m\pi}{\alpha}\phi'\sum_{n}\frac{\sinh(n+\frac{1}{2})\eta\sinh(n+\frac{1}{2})(c-\eta')}{\sinh(n+\frac{1}{2})c}\cdot\frac{\mathrm{S}_n^{\frac{m\pi}{\alpha}}(\mu)\mathrm{S}_n^{\frac{m\pi}{\alpha}}(\mu')}{\left[\frac{d}{dn}\mathrm{S}_n^{\frac{m\pi}{\alpha}}(\mu)\frac{d}{d\mu}\mathrm{S}_n^{\frac{m\pi}{\alpha}}(\mu)\right]_{\mu_0}^{\mu_1}}\quad\text{for }\eta<\eta'.\quad(27)$$

Dougall, *loc. cit.* § 35 (2) gives a different result.

A system of nonagons nonuply in perspective.

By Dr William P. Milne.

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In investigating the properties of the cubic curve C, we are led to consider its Hessian C′, its Cayleyan Γ and the Hessian of its Cayleyan Γ′. In the present paper we propose to deal with the intersections of the systems $C + \lambda C'$ and $\Gamma + \mu\Gamma'$.

We note that all the members of the system $C + \lambda C'$ have the same set of points and lines as flexes and harmonic lines respectively. Similarly the system $\Gamma + \mu\Gamma'$ has the same set of points and lines as harmonic points and cuspidal tangents respectively.

If we project any one of the flexes to infinity and take its harmonic line as the axis of x, every member of $C + \lambda C'$ has the form

$$uy^2 + v = 0$$

where u and v are polynomials in x of degrees 1 and 3 respectively. Furthermore, referred to the same axes $\Gamma + \mu\Gamma'$ has the tangential form

$$pm^2 + q = 0$$

where p and q are polynomials in l of degrees 1 and 3 respectively. Hence $C + \lambda C'$ and $\Gamma + \mu\Gamma'$ are each symmetrical about the axis of x. But $\Gamma + \mu\Gamma'$ is of degree 6; therefore, $C + \lambda C'$ intersects $\Gamma + \mu\Gamma'$ in 18 points, which divide themselves into two sets of 9, the images of one another in the axis of x. We thus obtain two nonagons in perspective, the flex at infinity on the axis of y being the centre of perspective. Dealing thus in succession with each flex and harmonic line, we see that $C + \lambda C'$ *intersects* $\Gamma + \mu\Gamma'$ *in two nonagons nonuply in perspective, the centres of perspective being the nine flexes.*

If we take $2\pi_1$, $2\pi_2$, $2\pi_3$ to be the periods of the Weierstrassian Elliptic Functions, the parameters of the flexes of the cubic $x = \wp(u)$, $y = \wp'(u)$ are as usual:

$$\begin{array}{ccc} 0 & \frac{2\pi_1}{3} & \frac{4\pi_1}{3} \\ \frac{2\pi_2}{3} & \frac{2\pi_1}{3} + \frac{2\pi_2}{3} & \frac{4\pi_1}{3} + \frac{2\pi_2}{3} \\ \frac{4\pi_2}{3} & \frac{2\pi_1}{3} + \frac{4\pi_2}{3} & \frac{4\pi_1}{3} + \frac{4\pi_2}{3} \end{array} \qquad (1)$$

u	$u+\frac{2\pi_1}{3}$	$u+\frac{4\pi_1}{3}$	$u+\frac{2\pi_2}{3}$	$u+\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$u+\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$u+\frac{4\pi_2}{3}$	$u+\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$u+\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$	FLEXES
$-u$	$-u-\frac{2\pi_1}{3}$	$-u-\frac{4\pi_1}{3}$	$-u-\frac{2\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\frac{4\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$	0
$-u-\frac{2\pi_1}{3}$	$-u-\frac{4\pi_1}{3}$	$-u$	$-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\frac{2\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\frac{4\pi_2}{3}$	$\frac{2\pi_1}{3}$
$-u-\frac{4\pi_1}{3}$	$-u$	$-u-\frac{2\pi_1}{3}$	$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\frac{2\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\frac{4\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$\frac{4\pi_1}{3}$
$-u-\frac{2\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\frac{4\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u$	$-u-\frac{2\pi_1}{3}$	$-u-\frac{4\pi_1}{3}$	$\frac{2\pi_2}{3}$
$-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\frac{2\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\frac{4\pi_2}{3}$	$-u-\frac{2\pi_1}{3}$	$-u-\frac{4\pi_1}{3}$	$-u$	$\frac{2\pi_1}{3}+\frac{2\pi_2}{3}$
$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\frac{2\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\frac{4\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\frac{4\pi_1}{3}$	$-u$	$-u-\frac{2\pi_1}{3}$	$\frac{4\pi_1}{3}+\frac{2\pi_2}{3}$
$-u-\frac{4\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u$	$-u-\frac{2\pi_1}{3}$	$-u-\frac{4\pi_1}{3}$	$-u-\frac{2\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$\frac{4\pi_2}{3}$
$-u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\frac{4\pi_2}{3}$	$-u-\frac{2\pi_1}{3}$	$-u-\frac{4\pi_1}{3}$	$-u$	$-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\frac{2\pi_2}{3}$	$\frac{2\pi_1}{3}+\frac{4\pi_2}{3}$
$-u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\frac{4\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\frac{4\pi_1}{3}$	$-u$	$-u-\frac{2\pi_1}{3}$	$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\frac{2\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$\frac{4\pi_1}{3}+\frac{4\pi_2}{3}$

(2)

u	$u+\frac{2\pi_1}{3}$	$u+\frac{4\pi_1}{3}$
$-u$	$-u-\frac{2\pi_1}{3}$	$-u-\frac{4\pi}{3}$
$-u-\frac{2\pi_1}{3}$	$-u-\frac{4\pi_1}{3}$	$-u$
$-u-\frac{4\pi_1}{3}$	$-u$	$-u-\frac{2\pi}{3}$
$-u-\frac{2\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+$
$-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\frac{2\pi}{3}$
$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right)$	$-u-\frac{2\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\right.$
$-u-\frac{4\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\right.$
$-u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\frac{4\pi}{3}$
$-u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$	$-u-\frac{4\pi_2}{3}$	$-u-\left(\frac{2\pi_1}{3}+\right.$

(2)

Now if we take u as the parameter of any point on $C + \lambda C'$ and join u to each of the flexes in turn, we get as one nonagon of the system the third points of intersection of these chords respectively

$$-u,\ -u-\frac{2\pi_1}{3},\ -u-\frac{4\pi_1}{3},\ -u-\frac{2\pi_2}{3},\ -u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right),$$

$$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right),\ -u-\frac{4\pi_2}{3},\ -u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right),\ -u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$$

The other nonagon with which the above is nonuply in perspective may easily be found by joining the above points to any one flex (say 0) and taking the points in which these chords cut the cubic again.

We obtain:

$$u,\ u+\frac{2\pi_1}{3},\ u+\frac{4\pi_1}{3},\ u+\frac{2\pi_2}{3},\ u+\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right),\ u+\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right),$$

$$u+\frac{4\pi_2}{3},\ u+\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right),\ u+\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right).$$

We may represent the collineations diagrammatically as follows: where a point in the upper line joined to a point in the same column passes through the flex in the same horizontal row. Thus $u+\frac{4\pi_2}{3}$ from the upper line joined to $-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$ from the same column passes through the flex $\frac{2\pi_1}{3}+\frac{4\pi_2}{3}$ in the same horizontal line as $-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$. (See subjoined diagram (2)).

The way in which the points of the above scheme replace one another may be more easily seen if we use the points themselves instead of their parameters. Thus let the flexes be

$$\begin{matrix} F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{matrix} \qquad (3)$$

instead of (1), where three flexes are collinear if the letters are the same (*e.g.* $G_1G_2G_3$), or if the suffixes are the same (*e.g.* $F_2G_2H_2$), or if the letters and suffixes are all different (*e.g.* $F_3G_2H_1$). The scheme (2) then becomes

P_1	P_2	P_3	Q_1	Q_2	Q_3	R_1	R_2	R_3	Flexes
X_1	X_2	X_3	Y_1	Y_2	Y_3	Z_1	Z_2	Z_3	F_1
X_2	X_3	X_1	Y_2	Y_3	Y_1	Z_2	Z_3	Z_1	F_2
X_3	X_1	X_2	Y_3	Y_1	Y_2	Z_3	Z_1	Z_2	F_3
Y_1	Y_2	Y_3	Z_1	Z_2	Z_3	X_1	X_2	X_3	G_1
Y_2	Y_3	Y_1	Z_2	Z_3	Z_1	X_2	X_3	X_1	G_2
Y_3	Y_1	Y_2	Z_3	Z_1	Z_2	X_3	X_1	X_2	G_3
Z_1	Z_2	Z_3	X_1	X_2	X_3	Y_1	Y_2	Y_3	H_1
Z_2	Z_3	Z_1	X_2	X_3	X_1	Y_2	Y_3	Y_1	H_2
Z_3	Z_1	Z_2	X_3	X_1	X_2	Y_3	Y_1	Y_2	H_3

(4)

If two points of the above system coincide, it is plain that all the others will coincide in pairs. For let the coincident pair not lie on the axis of x. Then it is plain that their images with respect to Ox will also coincide. If we use the property of symmetry with respect to all the harmonic lines as explained in the beginning of the paper, we see that we obtain nine pairs of coincident points. With respect to the particular harmonic line Ox, four of these coincident pairs must lie above Ox and four must be their images below. The ninth point must lie on Ox, and the tangent thereat to the cubic curve must be parallel to Oy, that is, must pass through the flex corresponding to that particular harmonic line. Thus, *when the eighteen points coincide in pairs, the tangent at each of the resulting nine points passes through one or other of the flexes. If the tangent at one of these points pass through the flex* F, *the other eight points lie two by two on four chords passing through* F.

We shall show that *the cross-ratio of the above four chords passing through* F *is equi-anharmonic.*

Let the point of contact of the tangent through F be T. Let p be a chord drawn from F, and let the two points in which p cuts the cubic be joined to T and cut the curve again. It can easily be shown that the chord p' joining these latter two points passes through F. Hence a one-to-one algebraic correspondence exists between p and p', and hence the cross-ratio of any four

positions of p is equal to that of the corresponding four positions of p'. Take the four positions of p on which lie the flexes other than F. We thus get a pencil of four lines whose cross-ratio is known to be equi-anharmonic. (5)

Let the flex F have 0 as its parameter.

Let T have as its parameter π_1.

Then if p_1 cut the cubic in the flexes $\frac{2\pi_1}{3}$ and $-\frac{2\pi_1}{3}$, plainly p_1' cuts the cubic in the points $-\pi_1-\frac{2\pi_1}{3}$ and $-\pi_1+\frac{2\pi_1}{3}$, and so on.

If we examine the scheme (2) and substitute throughout π_1 instead of u, we see that p_1', p_2', p_3', p_4' cut the cubic in the four pairs of coincident points required.

Hence $[p_1', p_2', p_3', p_4'] \equiv [p_1 p_2 p_3 p_4]$

$\equiv$ equi-anharmonic ratio (by (5)).

Examples of cases of two coincident nonagons are :—

"The Hessian touches the Cayleyan at the points where the inflexional tangents meet their respective harmonic lines."

If we isolate any one of the points in which the inflexional tangents meet their respective harmonic lines, the other eight lie two by two on four lines passing through that isolated flex, and the cross-ratio of this pencil is equi-anharmonic.

"The cusps of a class-cubic lie on a cubic curve having the same system of flexes and harmonic lines."

If we isolate any one of the cusps of a class-cubic, the other eight cusps lie two by two on four lines passing through the harmonic point of that isolated cuspidal tangent, and the cross-ratio of this pencil is equi-anharmonic.

If C be a nodal cubic, C′ is also a nodal cubic having the same nodal tangents and flexes. Again Γ and Γ' may be considered as a system of conics touching the nodal tangents at the points H and K, in which they meet the common line of flexes. Project H and K into the Circular Points at Infinity. We thus see that $\Gamma+\mu\Gamma'$ is a system of concentric circles, having the node as their common centre, cutting $C+\lambda C'$ in sets of six points forming the vertices of a hexagon consisting of two equilateral triangles. The Hessian touches the Cayleyan at three points forming the vertices of an equilateral triangle.

A Problem of Robert Simson's.

By Dr G. Philip.

(*Read 14th February 1913. Received 20th May 1913.*)

The following problem appears in Robert Simson's "Opera Quaedam Reliqua," pp. 472–504:

"Si a duobus punctis datis A, B ad circulum positione datum CDE inflectantur utcumque duae rectae AC, BC circumferentiae rursus in D, E occurrentes; juncta DE vel continebit datum angulum cum recta ad datum punctum vergente; vel parallela erit rectae positionae datae; vel verget ad datum punctum:" *i.e.* if from two given points A and B any two straight lines AC, BC are drawn to a circle CDE given in position, and they meet the circumference again in D and E, then the straight line DE (I.) will make a constant angle with a straight line passing through a fixed point, or (II.) will be parallel to a straight line given in position, or (III.) will pass through a given point. This final form of the result was only arrived at by Simson after he obtained the aid of Matthew Stewart.

As indicated, the discussion of the problem by Simson occupies over thirty pages quarto, but the proof may be given as follows:

Case I.(*a*). *When* AB *does not pass through the centre of the given circle.* This is the most general case. Find K (Fig. 1) on AB, such that $AB \cdot AK = AC \cdot AD$, then K is a definite point. Join S, the centre of the circle, to K, and find F the harmonic conjugate of K with respect to the circle; then $SF \cdot SK = ST^2$. Let AF meet the circle in L and M, then KL and KM cut off equal arcs MP and QL.

If BL be joined and cut the circle again in N, NP will be parallel to AB. (For $AK \cdot AB = AM \cdot AL$, and therefore LMKB is a cyclic quadrilateral. Hence $\angle AKM = \angle MLB = \angle MPN$). Similarly if CB cuts the circle in E, EH is parallel to AB. Hence $PN \parallel HE$, and therefore arc PH = arc EN. Again, if

DF meet the circle in G, KG and KD cut off equal arcs, *i.e.* arc HD = arc GR.

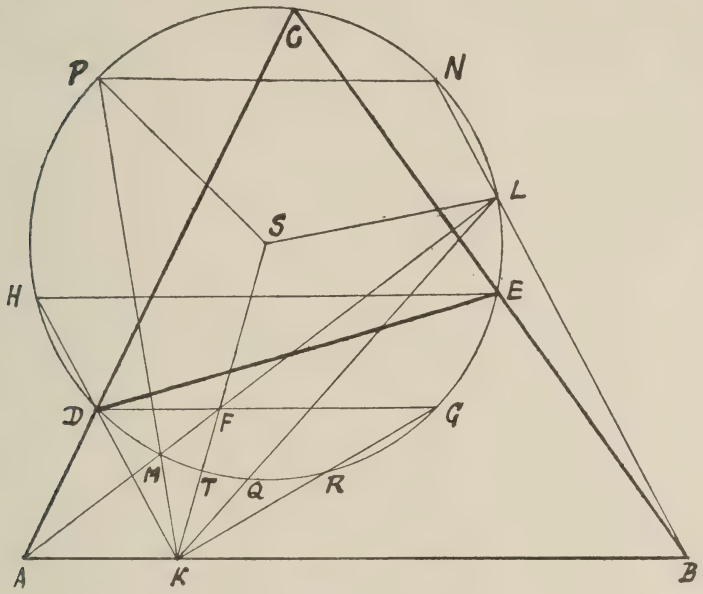

Fig. 1.

Now KP = KL (Pappus' Lemma), and therefore $\angle$ PSK = $\angle$ LSK, and arc TP = arc TL. Similarly, since arc HD = arc GR, KH = KG and arc TH = arc TG. Hence arc GL = arc PH = arc EN, and arc GE = arc NL. Thus

$$\angle \text{EDF or } \angle \text{EDG} = \angle \text{NML} = \text{constant.}$$

Case I.(*b*). *When* AB *passes through the centre.* Find K (Fig. 2) in AB such that AB . BK = CB . BE. Then since $\angle$ ABC = $\angle$ KBE, and the triangles ABC, KBE are thus similar, $\angle$ ACE = $\angle$ BKE. Then find F in AB such that SF . SK = ST^2. Let ES cut the circle in G, then SF . SK = SG^2 = SG . SE and $\angle$ GSF = $\angle$ KSE. Therefore the triangles GSF, KSE are similar, and $\angle$ FGS = $\angle$ SKE = $\angle$ ACE. If FG cuts the circle in D′ and AC in D, we have $\angle$ D′GE = $\angle$ DCE, wherefore arc ED′ = arc ED and D and D′ coincide. Hence FDG are collinear, and $\angle$ FDE is a right angle, and therefore constant.

Case II. When A *and* B *are inverse points with respect to the circle.* Let AB cut the circle in Y. Then $\angle$ DCY = $\angle$ YCE, and therefore Y is the middle point of the arc DE, and AB is perpendicular to DE. Thus DE is parallel to a fixed line.

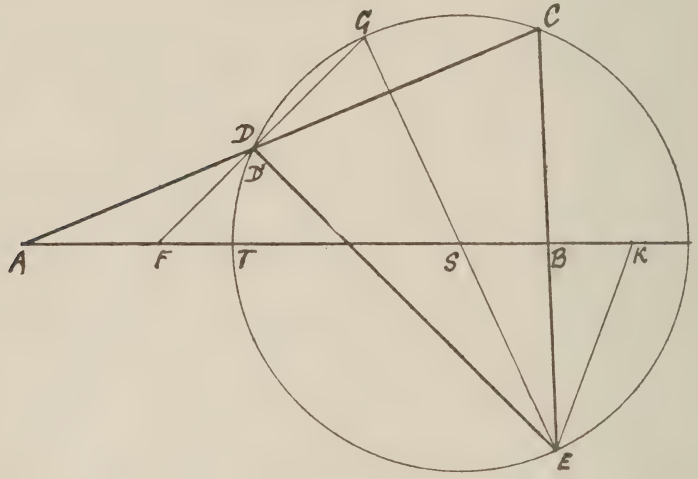

Fig. 2.

Case III. When A *and* B *are diagonal points of any cyclic quadrilateral* DCUE. Then DE passes through F the pole of AB with respect to the given circle, *i.e.* through a fixed point.

By reciprocation of Simson's result with respect to a circle whose centre is F, we obtain the following proposition: Let S be a conic with focus F, and a, b be two straight lines in the plane of the conic. If any tangent meet these lines in U and V, and if from these points tangents d, e be drawn to the conic and intersect at a point x, then the angle that Fx makes with either tangent d or e is constant.

A Geometrical Construction for the Rainbow Formula.

By Dr David Robertson.

(*Read 7th March 1913.*)

Internal Reflections in a Sphere. A ray of light PQ (Fig. 1) incident at any point Q on the surface of a transparent sphere is partly reflected and partly refracted along QR_1. At R_1 it is partly reflected along R_1R_2, and partly emerges along R_1S_1. The same thing occurs at R_2, R_3, etc., on which the successive reflected

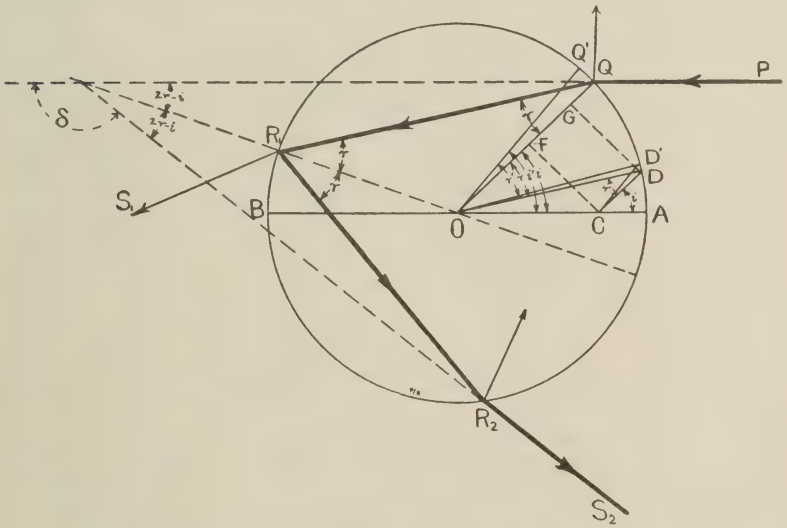

Fig. 1. *Internal Reflections in a Sphere.*

portions fall. AB is the diameter parallel to the incident light, and OQ the radius to the point of incidence. Let μ denote the index of refraction of the material of the sphere relative to the surrounding medium. Divide OA in C so that $OC : OA = 1 : \mu$.

Draw CD ‖ OQ and meeting the circle in D. Draw CF and DG ⊥ OQ.

Then, $\sin DOQ = \frac{GD}{OD} = \frac{FC}{\mu . OC} = \frac{1}{\mu}\sin COF = \frac{1}{\mu}\sin i = \sin r.$

$\therefore \quad \angle DOQ = r = \angle OQR.$

$\therefore$ DO is parallel to the ray refracted at Q.

The deviation of the ray is $(i - r)$ at incidence, and also at emergence; it is $(\pi - 2r)$ at each reflection, and so the total deviation with n reflections is :—

$$\begin{aligned}\delta &= 2(i - r) + n(\pi - 2r)\\ &= n\pi - 2\{(n+1)r - i\}.\end{aligned}$$

Minimum Deviation. Let Q move to an adjacent point Q′ further from A; then

$$\begin{aligned}\delta - \delta' &= -2\{(n+1)(r - r') - (i - i')\}\\ &= -2\{(n+1)(QOQ' - DOD') - QOQ'\}\\ &= -2\{nQOQ' - (n+1)DOD'\}\\ &= -2\{nDCD' - (n+1)DOD'\}\\ &= -2QOQ'\left\{n - (n+1)\frac{DOD'}{DCD'}\right\}\\ &= -2QOQ'\left\{n - (n+1)\frac{CD}{ODcosr}\right\}\\ &= -2QOQ'\left\{n - (n+1)\frac{CD}{OAcosr}\right\}.\end{aligned}$$

If Q be very close to A, this becomes

$$\begin{aligned}&-2QOQ'\left\{n - (n+1)\frac{CA}{OA}\right\}\\ &= -\frac{2}{\mu}QOQ'\left\{(n+1) - \mu\right\}.\end{aligned}$$

Since μ is less than 2, this last expression is always negative, which shows that the deviation at first diminishes as Q moves away from A, provided there be one or more internal reflections. It will continue to diminish with a further displacement of Q from

A until $(\delta - \delta')$ changes sign by passing through zero, after which it will increase. Hence, for minimum deviation

$$\delta - \delta' = 0$$

or
$$\mathrm{CD} = \frac{n}{n+1}\mathrm{OA}\cos r.$$

$$\frac{n}{n+1} = \frac{\mathrm{CD}}{\mathrm{OD}\cos r} = \frac{\sin(i-r)}{\sin i \,.\, \cos r}$$

$$= \frac{\sin i \,.\, \cos r - \cos i \,.\, \sin r}{\sin i \,.\, \cos r} = 1 - \frac{1}{\mu}\,\frac{\cos i}{\cos r}.$$

$$\therefore \quad \frac{\mu}{n+1} = \frac{\cos i}{\cos r}.$$

$$\therefore \quad (n+1)^2\cos^2 i = \mu^2\cos^2 r = \mu^2 - \mu^2\sin^2 r = \mu^2 - \sin^2 i$$

$$= \mu^2 - 1 + \cos^2 i.$$

$$\therefore \quad \cos i = \sqrt{\frac{\mu^2 - 1}{n(n+2)}}.$$

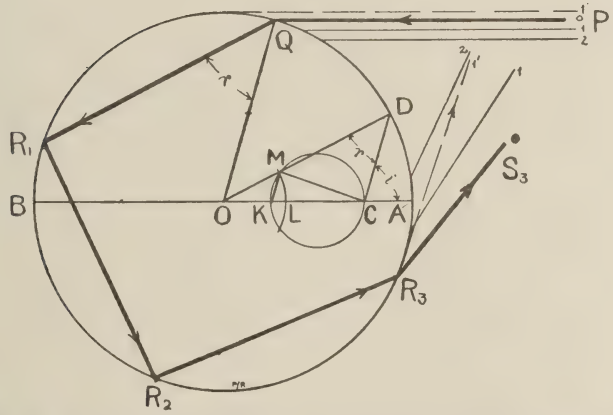

Fig. 2. *Minimum Deviation in a Sphere after two Reflections.*

Construction for Ray of Least Deviation. The large circle in Fig. 2 represents the sphere, and AB is the diameter parallel to the incident light.

Divide OA in C so that $\mathrm{OC} : \mathrm{OA} = 1 : \mu$.

„ OA in L „ $\mathrm{OL} : \mathrm{LA} = 1 : n$.

„ OC in K „ $\mathrm{OK} : \mathrm{KC} = 1 : n = \mathrm{OL} : \mathrm{LA}$.

Draw a circle on CK as diameter. Draw the arc LM with centre at O, and cutting the last circle at M. KM is parallel to the radius to the point of incidence, and OM is parallel to the direction after refraction of the ray which is at least deviated by the sphere after n internal reflections.

Proof. Continue OM to meet the outer circle at D, and join CD, CM, KM. Draw OQ ∥ CD. We have already proved that if OQ be parallel to CD, DO is the direction of the ray refracted at Q.

Then $OM : MD = OL : LA = OK : KC$,

$\therefore$ $KM \parallel CD \parallel OQ$, and $\therefore$ $CD \perp CM$.

Hence $CD = DM\cos CDM = LA\cos r$

$$= \frac{n}{n+1} AO\cos r,$$

which is the condition for minimum deviation.

Geometrical Constructions for Refraction and Reflection in a Prism.

By Dr David Robertson.

(*Read 7th March 1913*).

General Construction for Refracted Ray. The two-circle method of finding the direction of a ray refracted at a plane surface is very old, but seems to be now almost forgotten. It is particularly convenient when the refraction of several rays is to be determined, and its application to the case of a prism is especially elegant and leads to a simple self-contained proof of the condition for minimum deviation.

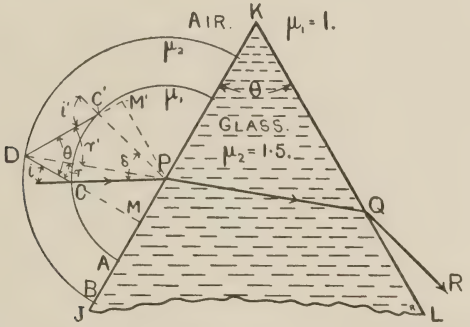

Fig. 1. *General Construction for Refracted Ray.*

In Fig. 1 let JKL be the prism, and P any point on its first surface. Draw two concentric circles whose radii represent to any convenient scale the indices of refraction of the material of the prism and of the surrounding medium, μ_2 and μ_1 respectively. The centre of these circles may with advantage be taken at P. The radius CP parallel to the incident ray cuts the circle for the first medium at C. If CD is drawn normal to the first surface, and cuts the circle for the second medium (prism) at D, then DP is the direction of the internal ray. If DC′ is drawn normal to the second surface, and cuts the circle for the outside medium at C′, then C′P is the direction of the emergent ray.

Proof. Draw PM $\perp$ DC.

Then $PD \cdot \sin PDM = PC \cdot \sin PCM.$

Or $$\sin PDM = \frac{PC}{PD} \sin PCM.$$

$$= \frac{\mu_1}{\mu_2} \sin i = \sin r.$$

$\therefore$ $\angle PDM = r$, the angle of refraction.

And DP is the direction of refraction.

A similar proof holds for the second refraction.

Deviation. CPC′ is the angle between the incident and emergent rays; that is, it is the deviation δ. CDC′ is the refracting angle of the prism θ, since CD and CD′ are perpendicular to the two faces.

But, $\theta = CDC' = CDP + C'DP = r + r'.$

And $$\begin{aligned} \delta = CPC' &= CPD + C'PD \\ &= (i - r) + (i' - r') \\ &= (i + i') - (r + r') \\ &= (i + i') - \theta. \end{aligned}$$

The angles i and r, or i' and r' are negative when the ray is deviated towards the thin end of the prism.

If $i = i'$, then $r = r'$, and

$$\delta = 2i - \theta$$

$$i = \tfrac{1}{2}(\delta + \theta)$$

and $$\frac{\mu_2}{\mu_1} = \frac{\sin i}{\sin r} = \frac{\sin \tfrac{1}{2}(\delta + \theta)}{\sin \tfrac{1}{2}\theta}.$$

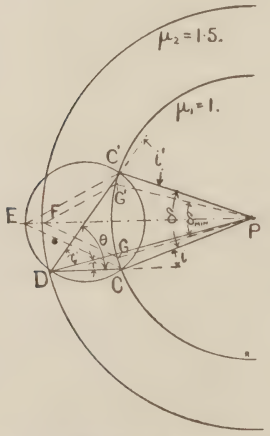

Fig. 2. *Deviation by a Prism.*

Minimum Deviation. Draw the circle through CDC′ (Fig. 2). Draw PE through the centre of this circle to cut it in E and the D-circle in F. Join EC, EC′. Draw FG ∥ EC and FG′ ∥ EC′, cutting the C-circle in G and G′. Then

$$GFG' = CEC' = CDC' = \theta = \text{constant}.$$

∴ arc GG′ and angle GPG′ are constant, since PF is a fixed length.

If D, E, and F all coincide the incidence and emergence are symmetrical, and the deviation is GPG′. With any other condition, E must lie outside the D-circle, and so C and C′ must lie outside GG′, and the deviation CPC′ must exceed GPG′. Consequently the deviation is least when the incidence and emergence are symmetrical; that is, when the internal ray is perpendicular to the bisector of the refracting angle.

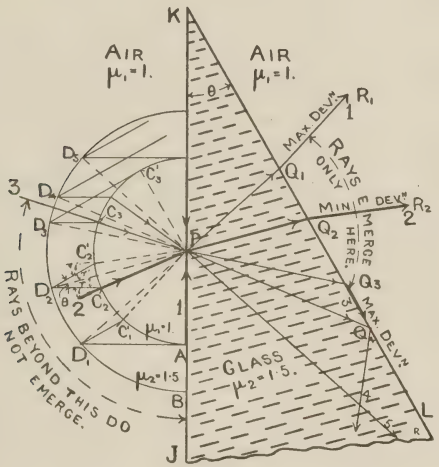

Fig. 3. *Emergent Rays with Fixed Prism.*

Limiting Rays. The ray numbered 1 in Fig. 3 enters the prism at grazing incidence, while 3 leaves it at grazing emergence. The latter is obtained by drawing PC_3' parallel to the second surface, and working the construction backwards from C_3'. It is obvious from the geometry of the diagram that the figures $AD_1C_1'P$ and $C_3'D_3C_3P$ could be made to coincide by suitably folding the diagram over, and that consequently the two limiting deviations are alike.

Rays beyond No. 3 do not emerge, but are totally reflected at the second surface.

Fig. 3 shows the emergent rays for a number of rays incident at the same point on a fixed prism.

Experimental Application. The following simple method of determining the index of refraction of a prism (due to Professor Poynting?*) gives good results, and is not generally known. Draw a straight line on paper, and place the prism over it in such a way

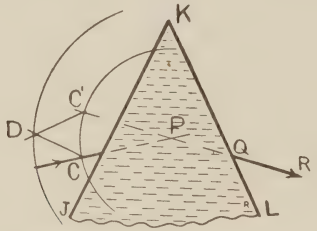

Fig. 4. *Experimental Determination of* μ.

it can be seen through the prism. Draw JKL (Fig. 4) the trace of the prism, and QR coinciding with the apparent direction of the line as seen through the prism. Remove the prism and produce RQ beyond P where it cuts the original line. With P as centre and any radius (conveniently a simple number of mms. or inches) draw an arc to cut the original line in C, and second line, produced backwards, in C′. Draw CD perpendicular to the surface of incidence JK. Draw C′D perpendicular to the surface of emergence KL, and intersecting CD in D. Then $\mu = DP \div CP$.

If a second circle, concentric with the first, be drawn through D, it will at once be recognised that the diagram is part of the two-circle construction already considered, and that it consequently gives the desired result. The only difference is that the centre of the circles has been moved to the point of intersection of the incident and emergent rays.

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* The author is indebted to Dr R. A. Houstoun for drawing his attention to this method.

A Simple Linkage for Describing Equal Areas.

By E. M. Horsburgh.

(*Read 13th June 1913. Received 22nd June 1913*).

If H (Fig. 1) be the middle point of a straight bar QP and if a straight bar OH of length one-half of QP be pin-jointed to QP at H, a simple linkage is formed, which may be called a ⊤-linkage.

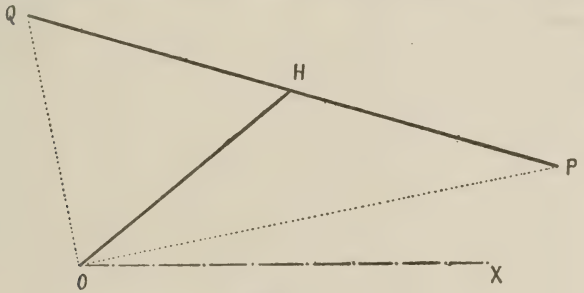

Fig. 1.

This, with an additional fixed bar guiding P or Q in a straight line through O, forms a well-known ellipsograph, or an equally well-known parallel motion. Let the ⊤-linkage be placed in the plane of the paper and let O be taken as pole, so that the linkage is free to turn about O in the plane of the paper. Then if P describe any closed curve not enclosing O, Q will describe a corresponding curve of equal area. For if the coordinates of P and Q be (r, θ) and (ρ, α) respectively, and if a be the length of OH, the equations of the transformation are

$$\text{(i)}\quad r^2+\rho^2=4a^2, \quad \text{(ii)}\quad \alpha=\frac{\pi}{2}+\theta.$$

Accordingly if P describe the curve $f(r, \theta)=0$, Q will describe the curve $f\left(\sqrt{4a^2-\rho^2},\ \alpha-\frac{\pi}{2}\right)=0$. Multiplying (i) by $\frac{1}{2}d\theta$ and integrating on the supposition that P describes a closed curve such that PQ returns to its original position without making a circuit about O, then $\frac{1}{2}\int r^2 d\theta+\frac{1}{2}\int \rho^2\delta\alpha=0$ taken round the curve, so that

if A_1 and A_2 be the areas described by P and Q respectively, $A_1 = -A_2$.

The equations of the transformed curves are most frequently complicated, but sometimes simple degenerate cases occur. Thus the family of straight lines $\theta = c$ transforms into the family of straight lines $\alpha = \frac{\pi}{2} + c$, and the family of concentric circles $r = k$ into the family of concentric circles $\rho = \sqrt{4a^2 - k^2}$, and thus the orthogonal network composed of the lines $\theta = c$ and the circles $r = k$ transforms into an equal area orthogonal network. (Figs. 2 and 3).

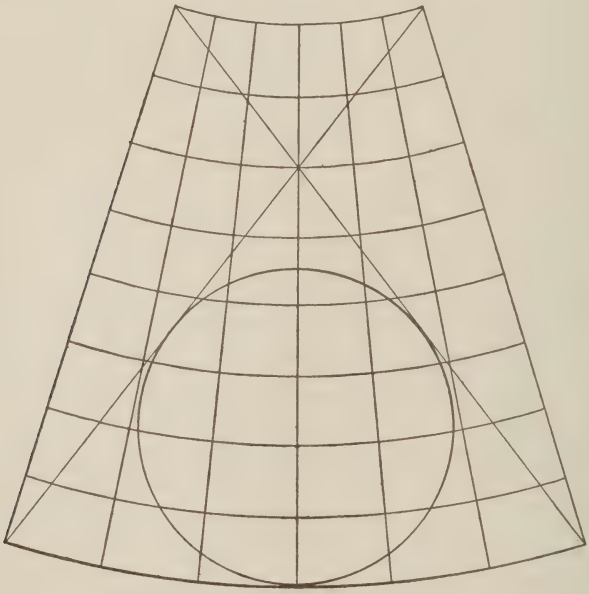

Fig. 2.

This suggested an application to some of the equal-area world-maps since the linkage deduces from any figure an infinite number of others all of equal area. Figs. 4, 5, 6, are representations of well-known equal-area world-maps, the first two by Lambert, the last by Collignon. Figs. 7, 8, 9, show equal-area maps deduced by the linkage and corrected with regard to the reversed

orientation which arises from the fact that P and Q describe figures in opposite senses.

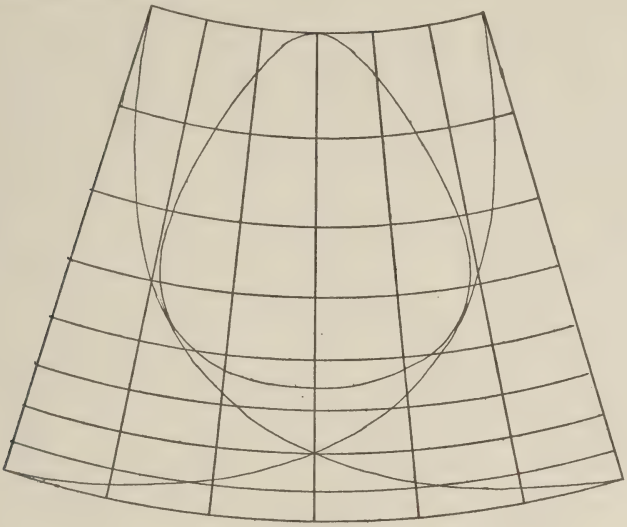

Fig. 3.

If the points P and Q describe arcs of curves instead of closed contours, then

$$\tfrac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta + \tfrac{1}{2}\int_{\alpha_1}^{\alpha_2} \rho^2 \delta\alpha = 2a^2(\theta_2 - \theta_1)$$

where θ_1, α_1 correspond to the initial, and θ_2, α_2 to the final positions of the tracing points. Thus the sum of the areas of the two sectors described by P and Q about the origin is equal to the area of the sector of a circle of radius PQ which contains the angle $\theta_2 - \theta_1$. If the linkage make a complete revolution about the pole O, PQ turns through an angle 2π, and the sum of the areas of the sectors traced by P and Q is equal to the area of a circle of radius PQ.

When one tracing point describes a curve passing through the pole an indeterminate condition arises, and a contour described twice by one tracing point may correspond to one described once by the other.

By taking a tracing point at any position S in PQ, an area can be drawn which is any required fraction of a given area A. If $QS = c$ and $SP = c'$, we have by a particular case of Holditch's Theorem that the area described by S is $A(c - c')/(c + c')$ when PQ rocks back to its original position.

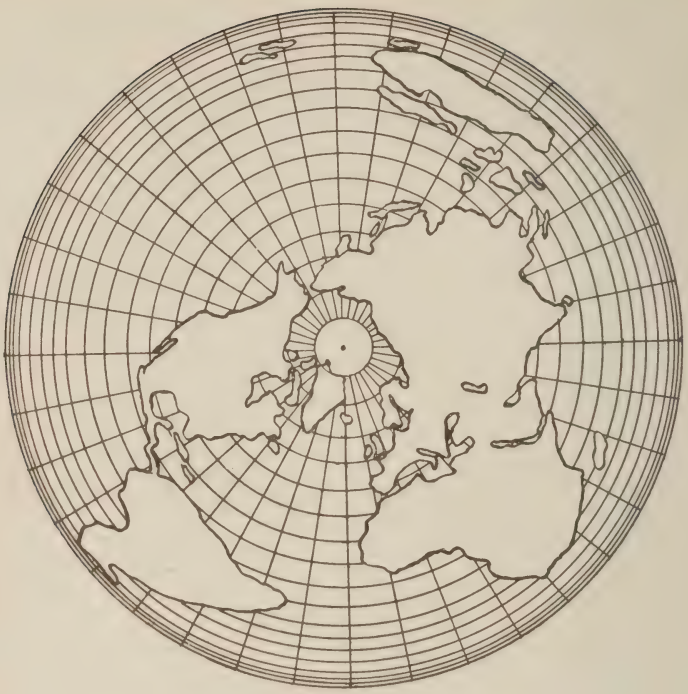

Fig. 4.

Fig. 5.

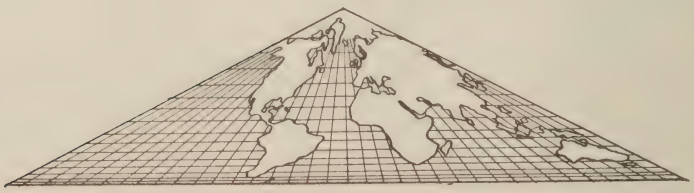

Fig. 6.

Fig. 7.

Fig. 8.

Fig. 9.

If (r, θ) and (ρ, α) be corresponding points on a curve and its transformed curve, and ϕ and Φ the angles between the radius vector and the tangent, then

$$\tan\phi = \frac{rd\theta}{dr}, \quad \tan\Phi = \frac{\rho d\alpha}{d\rho}.$$

Hence, since $rdr + \rho d\rho = 0$ and $d\alpha = d\theta$

$$\frac{\tan\phi}{\tan\Phi} = -\frac{r^2}{\rho^2}.$$

If ϕ' and Φ' be the angles for another corresponding pair of curves through these points, then

$$\frac{\tan\phi'}{\tan\Phi'} = -\frac{r^2}{\rho^2} = \frac{\tan\phi}{\tan\Phi},$$

which gives the law of intersection for the transformed network.

Edinburgh Mathematical Society.

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THIRTY-SECOND SESSION, 1913–1914.

First Meeting, Friday, 14th November 1913.

1. On a class of singularities discussed by Dr Sprague - A. B. Basset.

2. On the intersections of the six trisectors of each of the angles of any triangle - - - - - - F. G. Taylor and W. L. Marr.

Second Meeting, Friday, 12th December 1913.

1. Notes on Spherical Harmonics - - - - - - Dr John Dougall.

2. Napier's Logarithms - - - - - - - - Prof. G. A. Gibson.

3. Some results concerning homogeneous integral equations Pierre Humbert.

Third Meeting, Friday, 9th January 1914.

1. Note on the axes of a normal section of the enveloping cylinder of a conicoid - - - - - - - Dr R. J. T. Bell.

2. Notes on the equation of the Parabolic Cylinder - - Arch. Milne.

3. On the continued fractions which represent Hermite's functions and other functions defined by differential equations - - - - - - - - - Prof. E. T. Whittaker.

Fourth Meeting, Friday, 13th February 1914.

1. Concurrency of lines connected with a triangle - - A. G. Burgess.

2. Note on Gauss' proof of the reciprocity of parallelism Dr D. M. Y. Sommerville.

3. Description of a perpetual calendar - - - Dr D. M. Y. Sommerville.

4. On the quasi-periodic solutions of Mathieu's differential equation - - - - - - - - - - A. W. Young.

Fifth Meeting, Friday, 13th March 1914.

1. Some graphical methods for determining centres of gravity Dr R. F. Muirhead.

2. Some theorems analogous to Newton's theorems on symmetric functions, with elementary proofs - Dr R. F. Muirhead.

3. On certain formulae connected with the function $(\cos n\alpha - \cos n\theta)/(\cos\alpha - \cos\theta)$ - D. G. Taylor.

4. On summable series and Bernoullian numbers - - Prof. D. K. Picken.

Sixth Meeting, Friday, 8th May 1914.

1. Morley's Trisection Theorem :—
   (*a*) Its relation to the Hessian axis and the circumcentre — F. G. TAYLOR.
   (*b*) An extension and its relation to the circles of Apollonius - - - - - - - - - W. L. MARR.
2. On Simson's Line and related theory: an Exercise in Geometric Generality - - - - - - - Prof. D. K. PICKEN.
3. On a theorem in continued fractions equivalent to Riemann's Transformation of the P-function - E. LINDSAY INCE.

Seventh Meeting, Friday, 12th June 1914.

1. On the attraction of Spherical Shells, and a generalisation of the method for Ellipsoidal Shells - - - Prof. A. GRAY.
2. On the general solution of Mathieu's equation - Prof. E. T. WHITTAKER.
3. Napier's De Arte Logistica - - - - - - - Dr G. PHILIP.

The following Presents to the Library have been received, for which the Society tenders its grateful thanks :—

1. American Journal of Mathematics.
2. Bulletin of the American Mathematical Society.
3. Proceedings of the American Philosophical Society.
4. Annals of Mathematics.
5. Proceedings of the Cambridge Philosophical Society.
6. Proceedings of the Royal Society of Edinburgh.
7. Bulletin de la Société Mathématique de France.
8. Journal de Mathématiques Eléméntaires, publié par M. H. Vuibert.
9. Communication de la Société de Kharkow.
10. Proceedings of the London Mathematical Society.
11. Memoirs and Proceedings of the Manchester Literary and Philosophical Society.
12. The Mathematical Gazette.
13. Revista de la Real Academia de Ciencias exactas, fisicas y naturales de Madrid.
14. Annales de la Faculté des Sciences de Marseille.
15. The Journal of the Municipal School of Technology of Manchester.
16. Sitzungsberichte der Koniglich Bayerisehen Akademie der Wissenschaften Mathematische-Physikalische Krasse München.
17. Niew Archief voor Wiskunde.
18. Proceedings and Transactions of the Nova Scotian Institute of Science.
19. Rendiconti del Circolo Matematico di Palermo.
20. Annaes Scientificos de Academia Polytechnica do Porto.
21. Revue Semestrielle des Publications Mathématiques.
22. Proceedings of the Tôkyô Mathematico-Physical Society.
23. Annales de la Faculté des Sciences de l'Université de Toulouse.
24. Nova Acta regiae Societatis Scientiarum Upsaliensis.
25. Prace Matematyczno-Fizyczne, Warsaw.
26. Wiskundige Opgaven met de Oplossingen.
27. Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich.
28. Bollettino della "Mathesis" società italiana di matematica.
29. Journal of the Indian Mathematical Society.
30. Bulletin de la Societe Physico-Mathematique de Kasan.
31. Cambridge University Library Report.

32. Science Bulletin of the University of Kansas.

33. Bulletin of the International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics.

34. Science Reports of the Tôhoku Imperial University.

35. A Binary Canon; presented by Lt.-Col. ALLAN CUNNINGHAM, R.E.

36. L'Enseignement Mathématique.

37. Zur Projecktiveschen Behandlung der Dreiecksgeometrie; presented by GUSTAV BERKHAN.

38. Leipziger Zeitschrift für Deutsches Recht.

39. The Teaching of Mathematics in Australia; presented by Prof. H. S. CARSLAW.

40. Biography of T. J. J. Lee.

41. Lectures on the Icosahedron; FELIX KLEIN.

The following papers by Prof. Kelland have been presented by Miss Mackay:—

42. On a Process in the Differential Calculus.

43. On the Quantity of Light intercepted by a Grating Placed before a Lens.

44. On the Plane and Angle of Polarization of Light Reflected at the Surface of a Crystal.

45. On the Theory of Waves. Part II.

46. On the vibrations of an interrupted medium.

47. On General Differentiation. Part III.

48. On Molecular Equilibrium. Part I.

49. Reply to some Objections against the Theory of Molecular Action according to Newton's Law.

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On the Equation of the Parabolic Cylinder Functions.

By Arch. Milne, Research Student, Edinburgh University Mathematical Laboratory.

(*Read 9th January 1914. Received 29th January 1914*).

§1. *Introductory.*

Hermite, in 1864 (*Comptes Rendus*, vol. 58) introduced into analysis the polynomials defined by the relation

$$e^{-x^2}\mathrm{U}_n = \frac{d^n}{dx^n}\left(e^{-x^2}\right),$$

where n is a positive integer. He showed that they satisfied the differential equation

$$\frac{d^2\mathrm{U}_n}{dx^2} - 2x\frac{d\mathrm{U}_n}{dx} + 2n\mathrm{U}_n = 0,$$

that they were orthogonal functions, and that an arbitrary function $f(x)$ could be expanded in the form

$$f(x) = a_0\mathrm{U}_0 + a_1\mathrm{U}_1 + \ldots .$$

Five years later, Weber (*Math. Ann.*, vol. 1) in discussing the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

reduced it to the type

$$\frac{d^2 y}{dz^2} + \mathrm{Z}y = 0$$

where Z is a quadratic function of z, and showed that the functions of the parabolic cylinder were solutions of this equation. Various series which satisfy this equation have been found by Baer in 1883 (*Dissertation, Cüstrin*), and by Haentzschel in 1888 (*Zeitschrift für Mathematik*, vol. 33). In 1898, Markoff (*Bulletin de l'Academie de St Pétersbourg*, vol. 9) discusses the roots of the equation $e^{x^2}\frac{d^m}{dx^m}\left(e^{-x^2}\right) = 0$.

The next advance was made by E. T. Whittaker in 1902 (*Proc. London Math. Soc.*, vol. 35), who showed that Weber's functions were the same as Hermite's polynomials, and extended the latter to the case when n is not an integer. He solved the equation of the parabolic cylinder by means of a family of definite integrals, and showed that such previous results as Hermite's polynomials were but special cases of these. One of these definite integrals was defined as the standard solution of the equation, viz.,

$$D_n(z) = \frac{i\Gamma(n+1)}{2\pi} e^{-\frac{1}{4}z^2} z^n \int_\delta e^{-t-\frac{1}{2}(t^2/z^2)} (-t)^{-n-1} dt,$$

δ being a contour in the t-plane beginning and ending at infinity and encircling the origin. When the real part of n is negative, $D_n(z)$ can be expressed in the form

$$D_n(z) = \frac{1}{\Gamma(-n)} e^{-\frac{1}{4}z^2} z^n \int_0^\infty e^{-t-\frac{1}{2}(t^2/z^2)} t^{-n-1}\, dt.$$

The general solution of the differential equation of the parabolic cylinder was shown to be

$$aD_n(z) + bD_{-n-1}(iz)$$

where a and b are arbitrary constants. Recurrence formulae were also established, and the asymptotic expansion of $D_n(z)$ for large real positive values of z determined.

Adamoff, in 1906 (*Annales de l'Institut Polytechnique de St.-Pétersbourg*, vol. 5) discusses expansions of $D_n(z)$ when z is real and n is a large positive integer. In the following year, Myller-Lebedeff (*Math. Ann.*, vol. 64), in discussing integral equations, introduces the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0,$$

and determines as solutions a series of polynomials $V_n(x, t)$, which are homogeneous in x and $\sqrt{t}$, and which are intimately connected with the polynomials of Hermite.

In 1910, G. N. Watson (*Proc. London Math. Soc.*, Series 2, vol. 8) obtains the asymptotic expansion of $D_n(z)$ when n is large and real and z is complex. He evaluates certain contour-integrals involving $D_n(z)$, and determines the expansion of an arbitrary function in a series of the functions $D_n(z)$.

Lastly, H. E. J. Curzon, in 1913 (*Proc. London Math. Soc.*, Series 2, vol. 12) establishes the connection between the Hermite functions of the first and second kind, viz. $D_n(z)$ and $D_{-n-1}(iz)$, and the Legendre functions of the first and second kind. This is done by means of relations involving integrals analogous to those connecting the Bessel and Legendre functions.

§ 2. *Contents of Paper.*

In the present paper two independent series-solutions of the differential equation of the parabolic cylinder are expressed as definite integrals, and $D_n(z)$ expressed in terms of these by means of a linear relation of which the values of the constants are determined. Recurrence formulae connecting these constants are then established. Next, the solution of the parabolic cylinder equation is represented as the solution of a homogeneous integral equation. The auto-functions of this integral equation are then shown to be given by the functions $D_n(z)$. Lastly, certain values of $D_n(z)$ are given, which values were computed by means of the above relation involving $D_n(z)$ and the series-solutions of the equation; and, in addition, graphs illustrating $D_n(z)$ for different values of n are appended.

§ 3. *Series-solutions expressed as Definite Integrals.*

If we take Weber's differential equation for the parabolic cylinder, viz.,

$$\frac{d^2y}{dx^2} + Zy = 0$$

where Z is a quadratic function of z, and write it in the form adopted by Professor Whittaker, viz.,

$$\frac{d^2y}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

we get, as he points out, the two independent solutions

$$1 - \frac{2n+1}{4}z^2 + \frac{4n^2+4n+3}{96}z^4 - \ldots$$

and

$$z - \frac{2n+1}{12}z^3 + \frac{4n^2+4n+7}{480}z^5 - \ldots .$$

These two series adopt a more symmetrical form if we make the substitution

$$y = e^{-\frac{1}{4}z^2}u.$$

This effected, (1) becomes

$$\frac{d^2u}{dz^2} - z\frac{du}{dz} + nu = 0,$$

and the two series-solutions become

$$e^{-\frac{1}{4}z^2}\left[1 - \frac{n}{2\,!}z^2 + \frac{n(n-2)}{4\,!}z^4 - \frac{n(n-2)(n-4)}{6\,!}z^6 + \ldots\ldots\right]$$

and

$$e^{-\frac{1}{4}z^2}\left[z - \frac{n-1}{3\,!}z^3 + \frac{(n-1)(n-3)}{5\,!}z^5 - \frac{(n-1)(n-3)(n-5)}{7\,!}z^7 + \ldots\right],$$

which, for convenience, we shall denote by $\mathrm{E}_n(z)$ and $\mathrm{O}_n(z)$ respectively.

Considering $\mathrm{E}_n(z)$ first, we can represent this as a definite integral by expressing the various coefficients as Γ-functions.

Now

$$\int_0^\infty e^{-u}u^k\cos(cz\sqrt{u})\,du$$

$$= \int_0^\infty e^{-u}u^k\left[1 - \frac{c^2z^2u}{2\,!} + \frac{c^4z^4u^2}{4\,!} - \ldots\right]du$$

$$= \Gamma(k+1) - \Gamma(k+2)\frac{c^2z^2}{2\,!} + \Gamma(k+3)\frac{c^4z^4}{4\,!} - \ldots$$

$$= \Gamma(k+1)\left[1 - (k+1)c^2\frac{z^2}{2\,!} + (k+2)(k+1)c^4\frac{z^4}{4\,!} - \ldots\ldots\ldots\right].$$

If we compare this with the corresponding expansion in $\mathrm{E}_n(z)$, we readily find that, for equality, $c = \sqrt{-2}$ and $k = -\frac{n}{2} - 1$.

Hence we get

$$\int_0^\infty e^{-u}u^{-\frac{n}{2}-1}\cosh(z\sqrt{2u})du$$

$$= \Gamma\left(-\frac{n}{2}\right)\left[1 - \frac{nz^2}{2\,!} + \frac{n(n-2)}{4\,!}z^4 - \ldots..\right]$$

$$= \Gamma\left(-\frac{n}{2}\right)e^{\frac{1}{4}z^2}\mathrm{E}_n(z)$$

$$\therefore \quad \mathrm{E}_n(z) = \frac{1}{\Gamma\left(-\frac{n}{2}\right)} e^{-\frac{1}{4}z^2} \int_0^\infty e^{-u} u^{-\frac{n}{2}-1} \cosh(z\sqrt{2u}) du.$$

In an exactly similar fashion, by expanding $\sin(c'z\sqrt{u})$ in the integral

$$\int_0^\infty e^{-u} u^{k'} \sin(c'z\sqrt{u}) du,$$

and comparing the result with $\mathrm{O}_n(z)$, we find that

$$\mathrm{O}_n(z) = \frac{1}{\sqrt{2}\,\Gamma\left(-\frac{n-1}{2}\right)} e^{-\frac{1}{4}z^2} \int_0^\infty e^{-u} u^{-\frac{n}{2}-1} \sinh(z\sqrt{2u}) du.$$

§ 4. *Expression of $\mathrm{D}_n(z)$ in terms of $\mathrm{E}_n(z)$ and $\mathrm{O}_n(z)$.*

From the theory of linear differential equations there must be a linear relation connecting $\mathrm{D}_n(z)$—the standard solution of the differential equation in Professor Whittaker's paper—and $\mathrm{E}_n(z)$ and $\mathrm{O}_n(z)$ above, say,

$$\mathrm{D}_n(z) = a_n \mathrm{E}_n(z) + b_n \mathrm{O}_n(z),$$

where a_n and b_n are functions of n, but, of course, do not involve z. In order to get this relation, we note that

$$\left.\begin{aligned} \mathrm{E}_n(z) &= \frac{1}{\Gamma\left(-\frac{n}{2}\right)} e^{-\frac{1}{4}z^2} \mathrm{I}_\mathrm{E} \\ \mathrm{O}_n(z) &= \frac{1}{2^{\frac{1}{2}}\Gamma\left(-\frac{n-1}{2}\right)} e^{-\frac{1}{4}z^2} \mathrm{I}_\mathrm{O} \\ \mathrm{D}_n(z) &= \frac{1}{\Gamma(-n)} e^{-\frac{1}{4}z^2} z^n \mathrm{I}_\mathrm{D} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where

$$\mathrm{I}_\mathrm{E} = \int_0^\infty e^{-u} u^{-\frac{n}{2}-1} \cosh(z\sqrt{2u}) du$$

$$\mathrm{I}_\mathrm{O} = \int_0^\infty e^{-u} u^{-\frac{n}{2}-1} \sinh(z\sqrt{2u}) du$$

$$\mathrm{I}_\mathrm{D} = \int_0^\infty e^{-t-\frac{1}{2}(t^2/z^2)} t^{-n-1}\, dt.$$

If in I_E and I_O we substitute $t^2 = 2z^2u$, we get

$$I_E = 2^{\frac{n}{2}+1}\int_0^\infty e^{-\frac{1}{2}(t^2/z^2)}t^{-n-1}z^n\left[\tfrac{1}{2}(e^t + e^{-t})\right]dt$$

and

$$I_O = 2^{\frac{n}{2}+1}\int_0^\infty e^{-\frac{1}{2}(t^2/z^2)}t^{-n-1}z^n\left[\tfrac{1}{2}(e^t - e^{-t})\right]dt$$

whence

$$I_E - I_O = 2^{\frac{n}{2}+1}z^n\int_0^\infty e^{-t-\frac{1}{2}(t^2/z^2)}t^{-n-1}dt$$

$$= 2^{\frac{n}{2}+1}z^nI_D.$$

Using (2) above, we get the relation

$$D_n(z) = \frac{\Gamma\left(-\frac{n}{2}\right)}{2^{\frac{n}{2}+1}\Gamma(-n)}E_n(z) - \frac{\Gamma\left(-\frac{n-1}{2}\right)}{2^{\frac{n+1}{2}}\Gamma(-n)}O_n(z).$$

Putting $z = -\frac{n}{2}$ in the identity

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = \Gamma(2z)(2\pi)^{\frac{1}{2}}2^{\frac{1}{2}-2z}$$

we get the simpler form

$$D_n(z) = \frac{2^{\frac{n}{2}}\sqrt{\pi}}{\Gamma\left(-\frac{n-1}{2}\right)}E_n(z) - \frac{2^{\frac{n+1}{2}}\sqrt{\pi}}{\Gamma\left(-\frac{n}{2}\right)}O_n(z),$$

which gives us the desired relation.

§ 5. *Recurrence formulae connecting a_n and b_n.*

a_n and b_n are not independent, for if we substitute

$$a_nE_n(z) + b_nO_n(z)$$

for $D_n(z)$ in the recurrence formula

$$D_n(z) - zD_{n-1}(z) + (n-1)D_{n-2}(z) = 0$$

we get

$$a_nE_n(z) - zb_{n-1}O_{n-1}(z) + (n-1)a_{n-2}E_{n-2}(z)$$

and

$$b_nO_n(z) - za_{n-1}E_{n-1}(z) + (n-1)b_{n-2}O_{n-2}(z)$$

each equal to 0, the former being an even function of z and the latter an odd.

Substituting in these equations the values of $E_n(z)$ and $O_n(z)$, and equating the several coefficients to zero, we have these relations connecting a_n and b_n,

$$a_n + (n-1)a_{n-2} = 0$$
$$b_n + nb_{n-2} = 0$$
$$a_n + b_{n-1} = 0.$$

§ 6. *Expression of the solution of the parabolic-cylinder equation as the solution of a homogeneous integral equation.*

Reverting to the original form of the equation

$$\frac{d^2y}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)y = 0 \qquad (1)$$

if we assume

$$y = \int e^{izt}\phi dt$$

where ϕ is a function of t, we have

$$\frac{d^2y}{dz^2} = -\int e^{izt}t^2\phi dt.$$

Now

$$z^2y = \int e^{izt}z^2\phi dt$$
$$= \left[\frac{e^{izt}z\phi}{i}\right] + \int e^{izt}iz\frac{d\phi}{dt}dt$$
$$= \left[\frac{e^{izt}z\phi}{i}\right] + \left[e^{izt}\frac{d\phi}{dt}\right] - \int e^{izt}\frac{d^2\phi}{dt^2}dt.$$

Since the functions inside the square brackets can be made to vanish by suitable choice of limits, we have finally

$$z^2y = -\int e^{izt}\frac{d^2\phi}{dt^2}dt.$$

Substituting in (1), we get for that equation

$$\int e^{izt}\left[t^2\phi - (n+\tfrac{1}{2})\phi - \tfrac{1}{4}\frac{d^2\phi}{dt^2}\right]dt = 0$$

whence ϕ must satisfy the differential equation

$$\tfrac{1}{4}\frac{d^2\phi}{dt^2} + (n + \tfrac{1}{2} - t^2)\phi = 0,$$

or, s being substituted for $2t$, ϕ must satisfy

$$\frac{d^2\phi}{ds^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}s^2)\phi = 0,$$

which is obviously our initial equation (1).

If we call the solution of this last equation $\phi = y_n(s)$, then we get as the solution of (1)

$$y_n(z) = \lambda_n \int_a^b e^{\frac{izs}{2}} y_n(s) ds.$$

Limits of integration that satisfy the conditions are obviously $-\infty$ and $+\infty$, so that we have

$$y_n(z) = \lambda_n \int_{-\infty}^{\infty} e^{\frac{izs}{2}} y_n(s) ds.$$

§ 7. *Determination of* λ_n *in the integral equation by means of Fourier's Double-Integral Theorem.*

From the equation

$$y_n(z) = \lambda_n \int_{-\infty}^{+\infty} e^{\frac{izs}{2}} y_n(s) ds$$

we get

$$y_n(s) = \lambda_n \int_{-\infty}^{+\infty} e^{\frac{isu}{2}} y_n(u) du$$

$$\therefore \quad \frac{1}{\lambda_n^2} y_n(z) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} du\, e^{\frac{is}{2}(z+u)} y_n(u)$$

$$= \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} du \cos \frac{s(z+u)}{2} y_n(u)$$

$$+ i \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} du \sin \frac{s(z+u)}{2} y_n(u).$$

Now the former of these integrals is equal to

$$2 \int_0^{\infty} ds \int_{-\infty}^{\infty} du \cos \frac{s(z+u)}{2} y_n(u), \text{ since } \cos(-\theta) = \cos\theta$$

$$= 4 \int_0^{\infty} ds \int_{-\infty}^{\infty} du \cos s(z+u) y_n(u), \text{ on replacing } \frac{s}{2} \text{ by } s.$$

Similarly the latter of the integrals vanishes, since

$$\sin(-\theta) = -\sin\theta.$$

Hence

$$\frac{1}{\lambda_n^2} y_n(z) = 4 \int_0^{\infty} ds \int_{-\infty}^{\infty} du \cos s(z+u) y_n(u).$$

If we put $-z$ for z, this becomes

$$\frac{1}{\lambda_n^2} y_n(-z) = 4 \int_0^{\infty} ds \int_{-\infty}^{\infty} du \cos s(s-z) y_n(u)$$

$$= 4 \int_0^{\infty} ds \int_{-\infty}^{\infty} du \cos s(z-u) y_n(u).$$

Now from Fourier's Double-Integral Theorem,

$$\pi f(x) = \int_0^\infty d\mu \int_{-\infty}^{+\infty} f(\beta) \cos\mu(\beta - x) d\beta,$$

we get $$\int_0^\infty ds \int_{-\infty}^{\infty} du \cos s(z-u) y_n(u) = \pi y_n(z).$$

Hence $$\frac{1}{\lambda_n^{\,2}} y_n(-z) = 4\pi y_n(z).$$

But, as shown in the next paragraph, $y_n(z)$ is always either an odd or an even function of z and is of the form $e^{-\frac{1}{4}z^2} f_n(z)$ where $f_n(z)$ is of degree n in z. Therefore

$$y_n(-z) = (-1)^n y_n(z).$$

Hence $$\frac{1}{\lambda_n^{\,2}} = 4\pi(-1)^n, \text{ or } \lambda_n = \frac{1}{2\sqrt{\pi}(-1)^{\frac{n}{2}}},$$

so that the homogeneous integral equation may be written in the form

$$y_n(z) = \frac{1}{2\sqrt{\pi}(-1)^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{\frac{isz}{2}} y_n(s) ds.$$

§8. *Identity of $D_n(z)$ and $y_n(z)$.*

We next show that for integral values of n, $D_n(z)$ represents the auto-functions of the homogeneous integral equation

$$\frac{1}{\lambda_n} y_n(z) = \int_{-\infty}^{\infty} e^{\frac{isz}{2}} y_n(s) ds.$$

Consider the case where n is even and positive. Then since $D_n(z) = a_n E_n(z) + b_n O_n(z)$ (see §4), and in this case b_n vanishes, since $\Gamma(-p) = \infty$, it will be necessary and sufficient to show that

$$\frac{1}{\lambda_n} E_n(z) = \int_{-\infty}^{\infty} e^{\frac{isz}{2}} E_n(s) ds.$$

Now

$$\int_{-\infty}^{\infty} e^{\frac{isz}{2}} E_n(s) ds = 2e^{-\frac{1}{4}z^2} \int_{-\infty}^{\infty} e^{-x^2} \left[1 - \frac{n}{2!}(2x+iz)^2 + \frac{n(n-2)}{4!}(2x+iz)^4 - \ldots \right] dx$$

where $2x = s - iz$.

If we take as the general term z^r, where r of course is even, this will appear in the expansions of $(2x+iz)^r$, $(2x+iz)^{r+2}$, ... $(2x+iz)^n$. The terms involving z^r therefore are

$$4e^{-\frac{1}{4}z^2}\int_0^\infty e^{-x^2}\Big[(-1)^{\frac{r}{2}}\frac{n(n-2)\ldots(n-r+2)}{r!}(iz)^r$$

$$+(-1)^{\frac{r+2}{2}}{}_{r+2}C_2\frac{n(n-2)\ldots(n-r)}{(r+2)!}(iz)^r(2x)^2$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+(-1)^{\frac{n-2}{2}}{}_{n-2}C_{n-r-2}\frac{n(n-2)\ldots 4}{(n-2)!}(iz)^r(2x)^{n-r-2}$$

$$+(-1)^{\frac{n}{2}}{}_nC_{n-r}\frac{n(n-2)\ldots 2}{n!}(iz)^r(2x)^{n-r}\Big]dx.$$

Using the identity

$$\int_0^\infty e^{-x^2}x^{2n}dx=\frac{1\,.\,3\ldots(2n-1)}{2^{n+1}}\sqrt{\pi},$$

we find that the above is equal to

$$2\sqrt{\pi}(-1)^{\frac{r}{2}}(iz)^re^{-\frac{1}{4}z^2}\Big[\frac{n(n-2)\ldots(n-r+2)}{r!}-\frac{n(n-2)\ldots(n-r)}{r!}$$

$$+\frac{n(n-2)\ldots(n-r-2)}{r!\,2!}-\ldots\ldots\ldots$$

$$+(-1)^{\frac{n-r-2}{2}}\frac{n(n-2)\ldots 4}{r!\left(\frac{n-r-2}{2}\right)!}+(-1)^{\frac{n-r}{2}}\frac{n(n-r)\ldots 2}{r!\left(\frac{n-r}{2}\right)!}\Big]$$

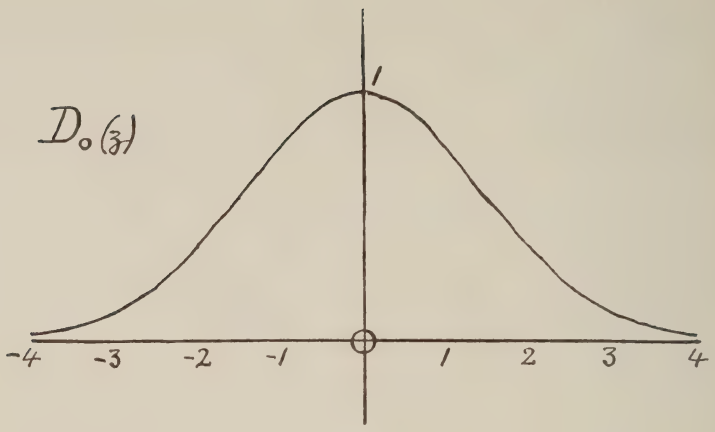
$$=\frac{2\sqrt{\pi}(-1)^{\frac{r}{2}}(iz)^re^{-\frac{1}{4}z^2}}{r!}n(n-2)\ldots(n-r+2)$$

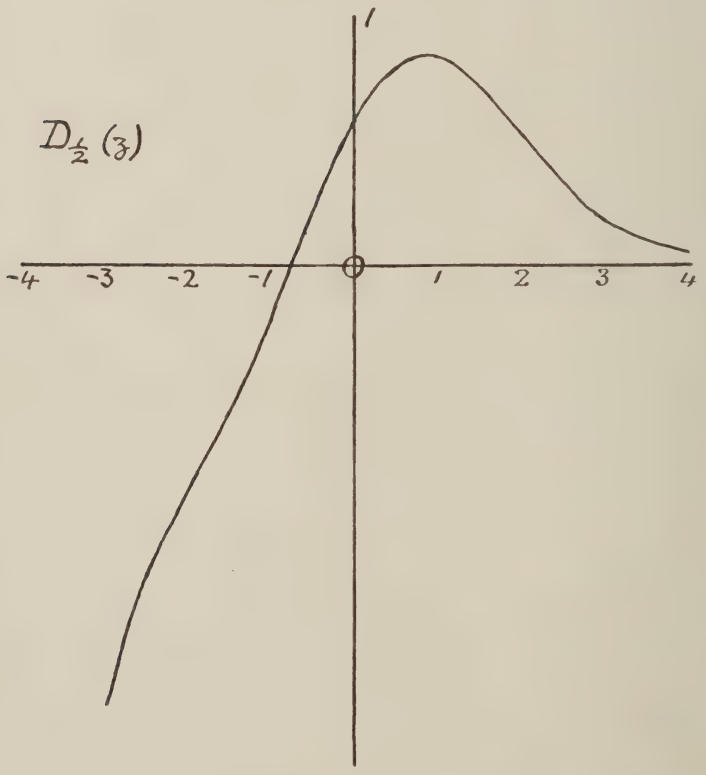
$$\Big[1-(n-r)+\frac{(n-r)(n-r-2)}{2!}-\ldots+(-1)^{\frac{n-r}{2}}\frac{(n-r)(n-r-2)\ldots 2}{\left(\frac{n-r}{2}\right)!}\Big].$$

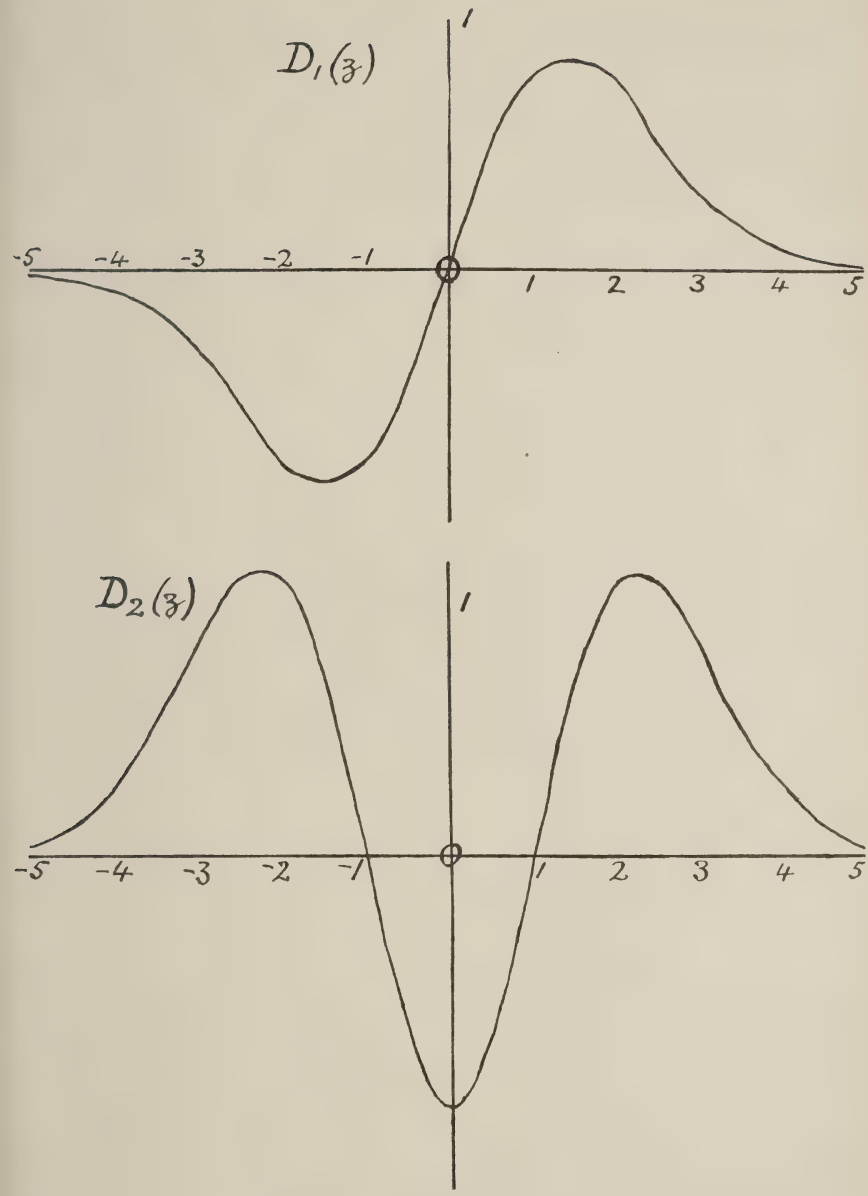
If we put $2p=n$ and $2q=r$, the expansion inside the brackets becomes

$$1-(p-q)\,.\,2+\frac{(p-q)(p-q-1)}{2!}\,.\,2^2-\ldots,$$

which is obviously the expansion of $(1-2)^{p-q}$, *i.e.* $(-1)^{\frac{n-r}{2}}$.


$D_0(z)$
1
-4
-3
-2
-1
1
2
3
4


$D_{\frac{1}{2}}(z)$
1
-4
-3
-2
-1
1
2
3
4


$D_1(z)$
1
-5
-4
-3
-2
-1
1
2
3
4
5
$D_2(z)$
1
-5
-4
-3
-2
-1
1
2
3
4
5

Hence we finally get

$$\int_{-\infty}^{\infty} e^{\frac{izs}{2}} \mathrm{E}_n(s)ds = 2\sqrt{\pi}\, e^{-\frac{1}{4}z^2} \sum_{r=0,\,2,\,\ldots n} (-1)^{\frac{n+r}{2}} \frac{n(n-2)\ldots(n-r+2)}{r\,!} z^r.$$

But this is equal to $\frac{1}{\lambda_n}\mathrm{E}_n(z)$, since $\frac{1}{\lambda_n} = 2\sqrt{\pi}(-1)^{\frac{n}{2}}$, and

$$\mathrm{E}_n(z) = e^{-\frac{1}{4}z^2}\left[1 - \frac{n}{2!}z^2 + \ldots + (-1)^{\frac{r}{2}} \frac{n(n-2)\ldots(n-r+2)}{r\,!} z^r + \ldots\right].$$

Parity of reasoning establishes the case when n is odd, and also when n is negative. Hence for integral values of n, the autofunctions of the integral equation are given by $\mathrm{D}_n(z)$.

§ 9. *Values of* $D_n(z)$ *when* $n = \frac{1}{2}$.

z	$\mathrm{D}_n(z)$	z	$\mathrm{D}_n(z)$
−3.0	−1.767855	0.1	0.639178
−2.5	−1.241135	0.2	0.690620
−2.0	−0.904956	0.3	0.735234
−1.5	−0.577806	0.4	0.772672
−1.0	−0.195001	0.5	0.802702
−0.9	−0.112590	0.6	0.825217
−0.8	−0.029289	0.7	0.840232
−0.7	0.054253	0.8	0.847884
−0.6	0.137314	0.9	0.848423
−0.5	0.219123	1.0	0.842203
−0.4	0.298877	1.5	0.727895
−0.3	0.375762	2.0	0.534014
−0.2	0.448977	2.5	0.337318
−0.1	0.517752	3.0	0.184882
0	0.581368		

Note on Gauss' Proof of the Reciprocity of Parallelism.

By Dr D. M. Y. Sommerville.

(*Read and Received 13th February 1914.*)

The proposition that if $AA' \parallel BB'$ then $BB' \parallel AA'$ appears at first sight so simple that it might be regarded as almost intuitive. This is because we already think of parallelism as a symmetrical relationship between two straight lines, in accordance with Euclid's definition of parallels as "straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction." If we take along with this definition Euclid's fifth postulate, or Playfair's equivalent, it defines a unique line through a given point parallel to a given line; but, without the postulate, it cannot be assumed to define more than a class of lines, and a stricter definition is required. Gauss' definition is as follows:

The straight line AA' is parallel to the straight line BB' when

(1) the two lines lie in the same plane;

(2) AA' does not cut BB';

(3) any ray through A within the angle BAA' cuts BB'.

This definition does not involve the two lines symmetrically, and it is not so obvious that any ray through B within the angle ABB' cuts AA', which is the reciprocal property that has to be proved.

It is possible to prove this by showing that a line can be drawn cutting AA' at P and BB' at Q, making equal angles $A'PQ$ and $B'QP$, and then making use of the symmetry of the "corresponding points" P, Q. The object of this note, however, is not to establish the theorem in any new way, but to examine two proofs which Gauss has given for the theorem.

In Vol. 8 of Gauss' Works, pp. 202–209, there are printed three notes, undated, but supposed to have been written in the

year 1831. No. 1 (pp. 202–206) is entitled "Parallellinien," and proves the three fundamental properties of parallelism:

(1) *Transmissibility, i.e.* if AA′ ∥ BB′ at the point A, it is ∥ BB′ at any other point in its length;

(2) *Reciprocity, i.e.* if $a \parallel b$, then $b \parallel a$.

(3) *Transitivity, i.e.* if $a \parallel b$ and $b \parallel c$, then $a \parallel c$.

No. 2 (p. 207) deals with corresponding points, and No. 3 (pp. 208–209) is entitled "Parallelismus," and deals with the same matters as the other two, but only in outline.

No. 3 contains the following proof of the Reciprocity Theorem: "Let 1 and 2 be parallel. If now 2 is not parallel to 1, let cd' be parallel to 1. Let ca be perpendicular to 1 and $acb = acb' = \frac{1}{2}dcd'$. Further $cbe = cb'b$. Therefore be will cut 2, in c'. Now make $b'g = be'$, then cg and cd' will make the same angle with cb'. Which is absurd."

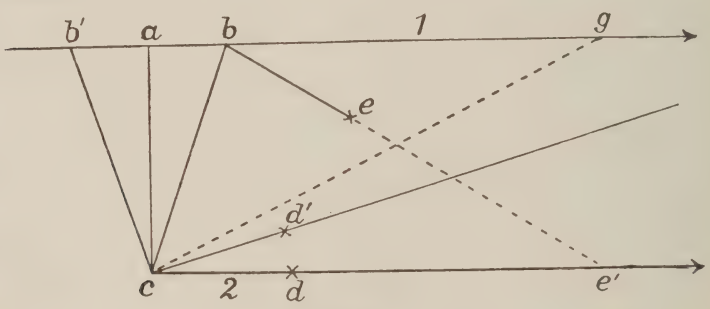


Fig. 1.

Now in this proof there is a fallacy, for it cannot be assumed, without assuming that $cd \parallel ab$, that the line cb which makes with ca an angle $= \frac{1}{2}dcd'$ will meet 1.

Bonola, in his book "La geometria non-euclidea," gives only a short resumé of Gauss' researches in the theory of parallels, and refers to this theorem in the single sentence, "this property [of reciprocity] forms the subject of another elegant demonstration of Gauss." Liebmann, in his German version of Bonola's book, changes this to "the following elegant demonstration of Gauss," and appends the above insufficient proof. The Russian translation follows Liebmann literally, but Professor Carslaw, in his English translation, has pointed out the fallacy, and gives instead the other proof by Gauss which is contained in No. 1.

This proof runs as follows*: "Let the straight line 1 be parallel to 2. From any point A in 2 let fall a perpendicular AB on 1. Let 3 be any straight line through A between AB and 2, and AC a straight line between the same boundaries, so that the angle $BAC = \frac{1}{2}(3, 2)$.

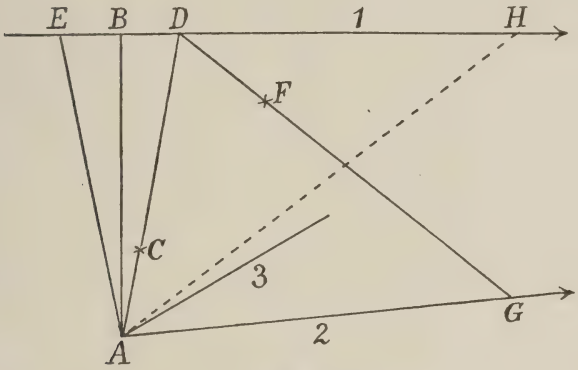


Fig. 2.

"We have now two cases to distinguish. (1) If AC...cuts the line 1 in D, make BE = BD, so that E is taken on 1 on the opposite side from D. Through D draw the straight line DF... between 1 and DA, so that ADF = AED. This straight line will therefore cut 2 in G. Make EH = DG and join AH. The triangles ABD, ABE will be congruent, therefore AE = AD; consequently also the triangles ADG, AEH congruent, therefore EAH = DAG. GAH = DAE = (2, 3), AH is identical with 3, or 3 cuts 1 in H, and consequently, since 3 may signify any line lying between 2 and AB, 2 is parallel to 1.

"(2) If AC does not cut 1, let D be any point on 1. There hold then the same conclusions as before up to the result GAH = DAE. Only in this case DAB < CAB or DAE < (2,3). Therefore (2, 3) > GAH and 3 will consequently lie in the closed figure AHD and therefore cut DH. The rest as in (1)."

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* Figs. 2 and 3 were added by the Editor. Note 1 was found apparently with no diagrams attached to it.

It is interesting to find the "giant mathematician" at fault, though our triumph is short-lived when we find the deficiency so amply made good in the alternative proof.

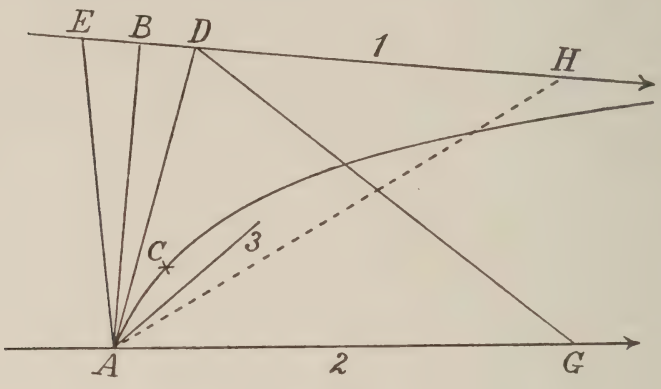

Fig. 3.

The occurrence of this fallacy proves almost conclusively, what is already indicated by the form of the three notes,* that in chronological order No. 3 is the first. Its character suggests that it consisted of rough jottings and outlines of what Gauss works out more elaborately in Nos. 1 and 2. It is almost inconceivable that he should have altered, even in a synopsis, a perfectly valid proof into an invalid one simply for the sake of simplification. Prof. Stäckel, who edited the geometrical part of this volume of Gauss' Works, and to whom I communicated my idea of the chronological order of these notes, admits that there is undoubtedly much to be said for this view. The question will be considered by him in a complete report on Gauss' geometrical works which he is preparing for the *Göttinger Nachrichten*, as a contribution to a scientific biography of Gauss. Similar reports on Gauss' works in the theory of numbers and function-theory have already been published by Bachmann in 1911 and Schlesinger in 1912.

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*For instance, in beginning the proof in No. 3 by the phrase, "Es sei 1 und 2 parallel," he implies the reciprocal relation that is to be proved.

On some results concerning integral equations.

By Pierre Humbert.

(*Read 12th December 1913. Received 23rd January 1914*).

1. *Description of the paper.*

It is proposed in this paper to show how the well-known Laplace's transformation,

$$y(z) = \int_0^b e^{azt} v(t) dt,$$

which is of great help in finding the solution of linear differential equations, gives also interesting results concerning the theory of integral equations. In §2 we shall study its application to certain differential equations, and find a large class of equations which remain unchanged by this transformation. Then, (§3), taking instead of e^{azt}, a more general function of the product zt, we shall find a solution for some homogeneous integral equations ; in § 4 we shall describe a method of solving a very general type of integral equation of the first kind, namely,

$$F(z) = \int_0^\infty f(zt)\Phi(t)dt \ ;$$

a further extension to integral equations with the kernel $e^{f(z)f(t)}$ is the object of §5. Then, studying an extension of Euler's transformation, we shall (§ 6) consider equations such as

$$y(z) = \int_{-\infty}^{+\infty} f(t-z)y(t)dt,$$

which will prove to be singular ; and finally, in § 7, we shall give other examples of singular integral equations.

2. *Laplace's transformation.*

Let us consider a linear differential equation of the n^{th} order

$$F(y, y', y'', \ldots, y^{(n)}, z) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and try to solve it by putting

$$y(z) = \int_a^b e^{azt} v(t) dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If the limits a and b are conveniently chosen, we obtain for v a linear differential equation of the n^{th} order,

$$G(v, v', v'', \ldots v^{(n)}, t) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Between a solution y_1 of (1) and a solution v_1 of (3) exists the relation (2), which is an integral equation of the first kind. If in particular we have

$$y_1 = \lambda_1 v_1$$

where λ_1 is a constant, y_1 will be a solution of the homogeneous integral equation

$$y(z) = \lambda \int_a^b e^{azt} y(t) dt$$

for the value λ_1 of λ. This case will always occur if, the equation (1) being of the first order, the equation (3) turns out to be precisely the same as (1).

If the two equations are the same, but of order n, then we shall have a relation such as

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = \lambda \int_a^b e^{at} [c_1' y_1 + c_2' y_2 + \ldots + c_n' y_n] dt,$$

which is an integral equation, generally of the first kind.

If we now consider the most general linear differential equation of order p, in which the coefficients are polynomials of degree p with respect to z, we can enquire what form these coefficients must have in order that they may remain unchanged by Laplace's transformation.

If we write the equation in the following form

$$\left.\begin{aligned} &(a_{p,p} z^p + a_{p,p-1} z^{p-1} + a_{p,p-2} z^{p-2} + \ldots + a_{p,0}) \frac{d^p y}{dz^p} \\ &+ (a_{p-1,p} z^p + a_{p-1,p-1} z^{p-1} + \ldots + a_{p-1,0}) \frac{d^{p-1} y}{dz^{p-1}} \\ &+ \ldots + (a_{0,p} z^p + a_{0,p-1} z^{p-1} + \ldots + a_{0,0}) y = 0, \end{aligned}\right\} \ldots\ldots\ldots (4)$$

it can be easily seen, by forming the coefficient of the term $t^n \frac{d^{p-j} v}{dt^{p-j}}$, in the transformed equation (n and j being arbitrary integers), that the equation will remain unchanged if, between the $(p+1)^2$ coefficients of (4) there exist the following $(p+1)^2$ relations,

where the symbol ${}_iC_r$ stands for the number of combinations of r numbers i at a time

$$\left.\begin{aligned} a_{p-j,\,n} = \alpha^{n+j-p}[&(-1)^p {}_jC_p (n+1)(n+2)\ldots(n+j)a_{n+j,\,p} \\ &+(-1)^{p-1}{}_{j-1}C_{p-1}(n+1)(n+2)\ldots(n+j-1)a_{n+j-1,\,p-1} \\ &+\ldots+(-1)^{p-j}{}_0C_{p-j}a_{n,\,p-j}] \end{aligned}\right\} (5)$$

where n and j take all the values 0, 1, 2, ...p,

With these formulae, it is possible to form the most general equation such as (4), of any order, which remains unaltered by Laplace's transformation.

Many remarks can be made from the formulae (5).

For instance, making

$$j=0 \quad n=p,$$

we obtain

$$a_{p,\,p} = (-1)^p a_{p,\,p}$$

which shows that, if p is an even number, $a_{p,\,p}$ can be taken arbitrarily. But, if p is an odd number, $a_{p,\,p}=0$.

In the same way, making

$$n=0 \quad j=p$$

we find that $a_{0,\,0}$ is always arbitrary.

These results give for the equation of the second order

$$(a_{2,\,2}z^2 + a_{2,\,0})y'' + 2a_{2,\,2}zy + (\alpha^2 a_{2,\,0}z^2 + a_{0,\,0})y = 0,$$

and for the equation of the third order,

$$\begin{aligned} &a_{3,\,1}zy''' + (a_{2,\,2}z^2 + a_{2,0})y'' \\ &\quad + (-\alpha^2 a_{3,\,1}z^3 + 2a_{2,\,2}z)y' + [\alpha^2 z^2(-3a_{3,\,1} + a_{2,\,0}) + a_{0,\,0}]y = 0. \end{aligned}$$

The above equation of the second order comprises, as a particular case, the well-known Euler's equation; it shows that our integral equation can be satisfied by the root $y(z) = \dfrac{1}{\sqrt{z}}$; but we do not insist now upon this result, which we shall soon consider in a more general case.

The case of the parabolic cylinder equation is also a particular case of this above equation; such is also the equation

$$y'' + z^2 y = 0,$$

whose solution can be expressed by the aid of Bessel's functions.

We shall now extend Laplace's transformation to integral equations with a more general kernel.

3. *Extension of Laplace's transformation.*

In the differential equation

$$2zy' + y = 0$$

let us make the following change of variable

$$y(z) = \int_a^b f(zt)v(t)dt$$

where f is an arbitrary function of the product zt. As we have

$$y'(z) = \int_a^b f'(zt)tv(t)dt,$$

our equation becomes

$$\int_a^b dt[2zf'(zt)tv(t) + f(zt)v(t)] = 0.$$

But we have, by integrating by parts,

$$2\int_a^b zf'(zt)tv(t)dt = \left[2f(zt)tv(t)\right]_a^b - 2\int_a^b f(zt)[tv' + v]dt.$$

Let us take now for a and b the values 0 and $\pm\infty$, if for these values

$$\left[2f(zt)tv(t)\right]_a^b = 0.$$

Then the differential equation which we obtain for v is

$$2tv' + v = 0,$$

and so is the same as the original equation for y. This equation having the solution $v = \frac{1}{\sqrt{t}}$, we find that the homogeneous integral equation

$$y(z) = \lambda \int_0^{\pm\infty} f(zt)y(t)dt$$

has, for a certain value of λ, the solution $y = \frac{1}{\sqrt{z}}$.

This value of λ is obtained by making $z = 1$, which gives

$$\frac{1}{\lambda} = \int_0^{\pm\infty} f(t)\frac{dt}{\sqrt{t}}.$$

By a change of variable, we find also that the integral equation

$$y(z) = \lambda \int_a^{\pm\infty} f[(z-a)(t-a)]y(t)dt$$

where $\frac{1}{\lambda} = \int_a^{\pm\infty} f[t-a]\frac{dt}{\sqrt{t-a}}$ has the solution $y = \frac{1}{\sqrt{z-a}}$.

4. *Application of the method to integral equations of the first kind.*

Instead of the above differential equation, let us now consider the following

$$zy' - ny = 0$$

whose solution is

$$y = z^n.$$

If we make the same change of variable, namely

$$y(z) = \int_0^\infty f(zt)v(t)dt,$$

we find in the same way that the transformed equation for v is

$$tv' + v(-n-1) = 0$$

which has the solution

$$v = t^{-1-n}.$$

So we find this result, that the homogeneous integral equation of the first kind

$$z^n = \int_0^\infty f(zt)v(t)dt$$

has the solution

$$v = \lambda_n t^{-1-n},$$

where λ_n is a constant determined by

$$\frac{1}{\lambda_n} = \int_0^\infty f(t)t^{-1-n}dt.$$

If we now consider an integral equation of the first kind

$$\mathrm{F}(z) = \int_0^\infty f(zt)\Phi(t)dt,$$

where F is known, and Φ unknown, we can find its solution in the following way:

Let us expand the function F(z), for points z within an annulus whose centre is at the origin, in a Laurent series, multiplied by a fractional power of z, if necessary, to allow for multiformity: so

$$\mathrm{F}(z) = z^h[a_0 + a_1z + a_2z^2 + \ldots + b_1z^{-1} + b_2z^{-2} + \ldots].$$

Then the unknown function $\Phi(t)$ will be

$$\Phi(t) = t^{-1-h}[\lambda_0a_0 + \lambda_1a_1t^{-1} + \lambda_2a_2t^{-2} + \ldots + \lambda_{-1}b_1t + \lambda_{-2}b_2t^2 + \ldots],$$

where the λ's are given by the formula

$$\lambda_i = \frac{1}{\int_0^\infty f(t)\,t^{-1-h-i}\,dt}.$$

As an example of this, we can take the integral equation

$$\frac{1+cz}{z^2} = \int_0^\infty e^{-\frac{1}{2}z^2t^2}\Phi(t)dt,$$

and it can be immediately seen that its solution, obtained by this method, is

$$\Phi(t) = t + c\sqrt{\frac{2}{\pi}}.$$

5. *Further extension.*

Let us consider now the equation

$$f(z)y' - \mathrm{A}f'(z)y = 0$$

where f is an arbitrary function of z, and make the transformation

$$y(z) = \int_a^b e^{f(z)f(t)}v(t)dt.$$

It becomes, after dividing by $f'(z)$,

$$\int_a^b dt\, e^{f(z)f(t)}[f(z)f(t)v(t) - \alpha v(t)].$$

On the other hand, we have

$$\int_a^b f(z)e^{f(z)f(t)}f(t)v(t)dt = \left[e^{f(z)f(t)}\frac{f(t)v(t)}{f'(t)}\right]_a^b - \int_a^b e^{f(z)f(t)}dt\frac{f'^2v + ff'v' - ff''v}{f'^2}$$

So that if we take for a and b two roots of $f(t) = 0$, the differential equation for v will be

$$f'(t)f(t)v' + v[(1+\mathrm{A})f'^2(t) - f(t)f''(t)] = 0.$$

This equation is not the same as the original one; and, if we choose the function f in order to make it the same, we shall find nothing but

$$f(z) = (z+m)^n,$$

which gives only a particular case of the kernel studied in § 2. However, we can go on with these differential equations. The solution of the equation for y is

$$y(z) = [f(z)]^{\mathrm{A}},$$

and for v,

$$v(t) = f'(t)\,[f(t)]^{-1-\mathrm{A}}.$$

So that we have the result that, a and b being two roots of $f(t)$, (or also values making $e^{f(t)}=0$, *i.e.* $f(t)=-\infty$), the homogeneous integral equation of the first kind

$$[f(z)]^{A}=\int_{a}^{b}e^{f(z)f(t)}v(t)dt$$

has the solution

$$v(t)=\lambda_{A}f'(t)\,[f(t)]^{-1-A}$$

where λ_{A} is a constant determined by

$$\frac{1}{\lambda_{A}}=\int_{a}^{b}e^{f(t)}f'(t)\,[f(t)]^{-1-A}dt.$$

We can extend this result, in the same way as in the preceding section, to an integral equation

$$\mathrm{F}[f(z)]=\int_{a}^{b}e^{f(z)f(t)}\Phi(t)dt$$

where F can be developed in powers of $f(z)$:

$$\mathrm{F}[f(z)]=a_{0}+a_{1}f(z)+a_{2}f^{2}(z)+\ldots.$$

We shall have then

$$\Phi(t)=f'(t)\left[\frac{a_{0}\lambda_{0}}{f(t)}+\frac{a_{1}\lambda_{1}}{f^{2}(t)}+\ldots\right].$$

the λ's being determined by the above formula.

6. *Other transformations.*

Euler has used the transformation

$$y(z)=\int_{a}^{b}(t-z)^{n}v(t)dt$$

to solve linear differential equations ; we can extend it, as we did Laplace's, to the theory of integral equations.

Let $f(u)$ be an even function of u, which is zero for $u=\pm\infty$; put

$$y(z)=\int_{-\infty}^{+\infty}f(t-z)v(t)dt.$$

The kernel is, as required in the ordinary theory, symmetrical, f being even. We have then

$$y'(z) = -\int_{-\infty}^{+\infty} f'(t-z)v(t)dt$$

$$= -\left[f(t-z)v(t)\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} f(t-z)v'(t)dt$$

$$= \int_{-\infty}^{+\infty} f(t-z)v'(t)dt.$$

so that, if we consider the equation

$$y' - ay = 0,$$

it becomes, by this change of variable,

$$v' - av = 0.$$

The solution of this equation being $v = e^{at}$, we find in that way that the homogeneous integral equation of the second kind

$$y(z) = \int_{-\infty}^{+\infty} f(t-z)y(t)dt$$

has, for a certain value of λ, the solution $y = e^{az}$.

The question is now, what is this value of λ.

We obtain it as we did before, and find

$$\frac{1}{\lambda} = \int_{-\infty}^{+\infty} f(t)e^{at}dt.$$

But a is an arbitrary constant, to which we can give an infinity of values; to each of these values corresponds a value of λ, and inversely, to a given value of λ corresponds a quantity a, and therefore a solution e^{az} for the integral equation. This equation appears then to be a singular integral equation, not having, as in the general case, a limited number of *auto-values* for the parameter λ, but having an infinity of them; in fact, any given number is an auto-value. The relation between the given λ and the corresponding a is the above. It can also be pointed out that to a chosen λ correspond two values of a, namely a and $-a$, for

$$\int_{-\infty}^{+\infty} e^{at}f(t)dt = \int_{-\infty}^{+\infty} e^{-at}f(t)dt.$$

But these are not the only singularities of that equation. The special value of λ corresponding to $a = 0$, *i.e.* λ_0, where

$$\frac{1}{\lambda_0} = \int_{-\infty}^{+\infty} f(t)dt = 2\int_0^{\infty} f(t)dt$$

is remarkable in another way. For if we consider the integral

$$I = \int_{-\infty}^{+\infty} f(t-z)tdt,$$

we can, to find its value, make the change of variable $t = u + z$, so that

$$I = \int_{-\infty}^{+\infty} f(u)udu + z\int_{-\infty}^{+\infty} f(u)du = \frac{z}{\lambda_0},$$

for, owing to the fact of f being even, the first of these two integrals is equal to nothing. So, for the particular value $\lambda = \lambda_0$, our equation is satisfied not only by $y(z) = \text{const.}$, but also by $y(z) = z$. It can be shown in a similar way that if the integral

$$\int_{-\infty}^{+\infty} f(u)u^2du$$

is equal to nothing, the equation possesses also the root $y(z) = z^2$, for the same value of λ. And then, in this case, as

$$\int_{-\infty}^{+\infty} f(u)u^3du = 0,$$

it has also the root $y(z) = z^3$, and so on.

Remarks of the same kind can be made on the subject of the integral equation

$$y(z) = \lambda\int_{-\infty}^{+\infty} f(z+t)y(t)dt,$$

which can be connected with the differential equation

$$y'' + ay = 0,$$

and is a singular integral equation, any value of λ being an auto-value; and the two values

$$\frac{1}{\lambda} = \pm\int_{-\infty}^{+\infty} f(u)du$$

having special properties.

We can also extend our method to integral equations of the type

$$y(z) = \int F[f(z) - f(t)]v(t)dt,$$

with convenient limits; they can be connected to the equation

$$y' - af'(z)y = 0,$$

and we obtain the integral equation of the first kind

$$e^{af(z)} = \lambda \int \mathrm{F}[f(z) - f(t)] f'(t)\, e^{af(t)} dt.$$

As we did in some of the preceding sections, we can apply this result to the case of a function such as

$$y(z) = a_0 + a_1 e^{f(z)} + a_2 e^{2f(z)} + \ldots .$$

But this transformation, as well as the similar one,

$$y(z) = \int \mathrm{F}[f(z) + f(t)] v(t) dt,$$

is of no use to obtain roots for new types of integral equations of the second kind.

7. *Singular integral equations.*

This brings us to say some few words about homogeneous singular integral equations. It is well known that the equation

$$y(z) = \lambda \int_0^\infty \cos zt \,.\, y(t) dt$$

has, for the special value $\lambda = \sqrt{\frac{2}{\pi}}$, an infinite number of solutions, owing to the fact that the integral

$$f(z) = \int_0^\infty \cos zt \phi(t) dt$$

possesses an inversion formula, which is Fourier's formula,

$$\phi(z) = \frac{2}{\pi} \int_0^\infty \cos zt f(t) dt.$$

We can extend this as follows:

Let us consider the integral equation

$$f(z) = \int_a^b \mathrm{K}(z, t) g(t) \phi(t) dt,$$

where $\mathrm{K}(z, t)$ is a symmetrical kernel, $g(t)$ a known function, and $\phi(t)$ an unknown one, and suppose that an inversion formula exists for this integral, and is of the type

$$\phi(z) = \mathrm{A} \int_a^b \mathrm{K}(z, t) g(t) f(t) dt,$$

where A is a constant. (Fourier's formula is a particular case of this inversion).

Then let us consider the function

$$\psi(z) = f(z) + \frac{1}{\sqrt{A}}\phi(z).$$

Replacing f and ϕ by their expressions as integrals, we have

$$\psi(z) = \int_a^b dt K(z, t)g(t)\left[\phi(t) + \sqrt{A}\, f(t)\right] = \sqrt{A}\int_a^b dt K(z, t)g(t)\psi(t).$$

We obtain thus an integral equation of the first kind. Let us make the change of variable

$$t = \chi(\theta)$$

where θ is defined by

$$\theta = \int g(t)dt.$$

Let us put also $z = \chi(u)$, and denote $\psi[\chi(u)]$ by $\Psi(u)$. Our integral equation becomes an integral equation of the second kind,

$$\Psi(u) = \sqrt{A}\int_a^b K[\chi(u), \chi(\theta)]\Psi(\theta)d\theta,$$

and, for this special value $\sqrt{A}$ of the parameter, it has an infinite number of roots, namely,

$$\Psi(u) = f[\chi(u)] + \frac{1}{\sqrt{A}}\phi[\chi(u)].$$

A remarkable example of this happening is given by using Hankel's formula

$$f(z) = \int_0^\infty J_n(zt)t\phi(t)dt, \quad \phi(z) = \int_0^\infty J_n(zt)tf(t)dt\,;$$

it shows that the integral equation

$$\psi(z) = \int_0^\infty J_n(2\sqrt{zt})\psi(t)dt$$

has an infinite number of solutions.

Notes on Spherical Harmonics.

By John Dougall, M.A., D.Sc.

(*Read 12th December 1913. Received 30th March 1914*).

This paper contains

(*a*) an explicit solution of the problem of finding a function which is harmonic within a given sphere and takes at the surface the same value as a given rational integral homogeneous function of the rectangular coordinates of a point referred to the centre of the sphere as origin;

(*b*) a concise symbolical expression for the integral, over the surface of the sphere, of the product of any three rational integral spherical harmonics.

I believe that the results are new. The expression, for *two* spherical harmonics, corresponding to that of (*b*) has been given by Cayley.*

1. Write as usual

$$r^2 \equiv x^2 + y^2 + z^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

If u_n is a homogeneous function of x, y, z of degree n, it is easy to prove that

$$\nabla^2(r^p u_n) = p(p+2n+1)r^{p-2}u_n + r^p\nabla^2 u_n, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and thence that, if n is a positive integer, the function defined by the finite series

$$u_n - \frac{1}{2(2n-1)}r^2\nabla^2 u_n + \frac{1}{2\,.\,4(2n-1)\,(2n-3)}r^4\nabla^4 u_n - \ldots\ldots\ldots(4)$$

is a rational integral spherical harmonic of degree n, which we may denote by $\mathrm{H}(u_n)$. From (4) it is obvious that, if $\mathrm{H}(u_n)$ vanishes, u_n contains r^2 as a factor.

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* *Cf.* a paper "On a certain expression for a spherical harmonic, with some extensions," *Proc. Edin. Math. Soc.*, Vol. VIII.

Interesting examples of (4) are obtained by taking

$$u_n = z^n \quad \ldots\ldots (5)$$

$$\left.\begin{aligned} u_n &= (x^2+y^2)^{\frac{n}{2}}, \quad n \text{ even}, \\ u_n &= (x^2+y^2)^{\frac{n-1}{2}} z, \quad n \text{ odd}, \end{aligned}\right\} \quad \ldots\ldots (6)$$

$$u_n = (x+iy)^m z^{n-m}. \quad \ldots\ldots (7)$$

With regard to (7) it is worth noting that the series (4) will terminate for any value of n, provided $n-m$ is a positive integer.

2. It is well known, and will be proved immediately, that u_n can be expanded in a finite series of the form

$$u_n = \mathrm{Y}_n + r^2\mathrm{Y}_{n-2} + r^4\mathrm{Y}_{n-4} + \ldots\ldots, \quad (8)$$

where Y_n, Y_{n-2}, ... are harmonic functions of the degrees indicated by the suffixes. The usual way of determining Y_n, Y_{n-2}, ... is as follows. By successive applications of ∇^2, we get from (8) with the help of (3)

$$\nabla^2 u_n = 2(2n-1)\mathrm{Y}_{n-2} + 4(2n-3)r^2\mathrm{Y}_{n-4} + \ldots\ldots, \quad (9)$$

$$\nabla^4 u_n = \quad 2\,.\,4(2n-3)(2n-5)\mathrm{Y}_{n-4} + \ldots\ldots, \quad (10)$$

and so on.

The last of such equations will give the lowest harmonic Y explicitly; then the last equation but one will give the lowest harmonic Y but one; and so on. But it does not seem to have been noticed that a general formula for any of the harmonics Y can easily be found. For it is obvious that an identity exists of the form

$$u_n = \mathrm{H}(u_n) + \mathrm{C}_2 r^2 \mathrm{H}(\nabla^2 u_n) + \mathrm{C}_4 r^4 \mathrm{H}(\nabla^4 u_n) + \ldots\ldots, \quad (11)$$

where C_2, C_4, ... are constants.

This is easily seen if we write down a few terms in full, say

$$\left.\begin{aligned} u_n = u_n &- \frac{1}{2(2n-1)} r^2\nabla^2 u_n + \frac{1}{2\,.\,4(2n-1)(2n-3)} r^4\nabla^4 u_n - \ldots \\ &+ \frac{r^2}{2(2n-1)}\left\{\nabla^2 u_n - \frac{1}{2(2n-5)} r^2\nabla^4 u_n + \ldots\right\} \\ &+ \frac{r^4}{2\,.\,4(2n-3)(2n-5)}\left\{\nabla^4 u_n - \ldots\right\} \end{aligned}\right\} \quad (12)$$

Here the coefficient of r^2 in the second line is taken so that the term in $r^2\nabla^2 u_n$ may disappear; then the coefficient of r^4 in the third line so that the term in $r^4\nabla^4 u_n$ may disappear; and so on.

But, once the expansion (11) is seen to be possible, the coefficients C_2, C_4, ... are easily obtained thus. Operate on both sides of (11) by ∇^2, ∇^4, ... Then, as at (9), (10)

$$\nabla^2 u_n = C_2 . 2(2n-1)H(\nabla^2 u_n) + C_4 . 4(2n-3)r^2 H(\nabla^4 u_n) + ..., \quad (13)$$

$$\nabla^4 u_n = \quad C_4 . 2.4(2n-3)(2n-5)H(\nabla^4 u_n) + ..., \quad (14)$$

and so on.

Now the process described at (9), (10) shows that only one expansion of the form (8) is possible. And (11) then shows that the highest harmonic in the expansion of $\nabla^2 u_u$ in the form (8) is $H(\nabla^2 u_n)$. Thus from (13)

$$C_2 . 2(2n-1) = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

Similarly, $$C_4 . 2 . 4(2n-3)(2n-5) = 1, \ldots\ldots\ldots\ldots\ldots \quad (16)$$

and, generally,

$$C_{2p} . (2 . 4 \ldots 2p)(2n-2p+1)(2n-2p-1)\ldots(2n-4p+3) = 1 \ldots \quad (17)$$

The values of u_n in (5), (6), (7) give useful examples of (11). The solution of the problem stated in (*a*) follows easily, for it is seen at once from (11) that the function which is harmonic within the sphere $r = a$, and takes the same value as u_n at the surface, is

$$H(u_n) + C_2 a^2 H(\nabla^2 u_n) + C_4 a^4 H(\nabla^4 u_n) + \ldots\ldots\ldots \quad (18)$$

3. Since the surface integral of any rational integral harmonic of degree greater than zero vanishes when taken over the surface of a sphere with centre at the origin, it follows that the surface integral of the function u_n depends only on the constant term, or harmonic of degree zero, in the expansion (11).

If n is odd, this harmonic does not occur, and the surface integral vanishes.

If n is even, the last term of (11) is

$$\frac{1}{(n+1)!} r^n \nabla^n u_n,$$

and in this case, if $n = 2p$, and the radius of the sphere is a,

$$\iint u_{2p} dS_a = \frac{4\pi}{(2p+1)!} a^{2p+2} \nabla^{2p} u_{2p} \ldots\ldots\ldots\ldots\ldots \quad (19)$$

4. For an application of (19) take

$$u_{2p} = (a_1 x + b_1 y + c_1 z)^l (a_2 x + b_2 y + c_2 z)^m,$$

where

$$\left.\begin{aligned} a_1^2 + b_1^2 + c_1^2 &= 0, \\ a_2^2 + b_2^2 + c_2^2 &= 0. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

As a consequence of (18) the functions $(a_1x+b_1y+c_1z)^l$ and $(a_2x+b_2y+c_2z)^m$ are both harmonic.

Hence

$$\nabla^2 u_{2p} = 2lm(a_1a_2+b_1b_2+c_1c_2)\,(a_1x+b_1y+c_1z)^{l-1}(a_2x+b_2y+c_2z)^{m-1},$$

and, continuing the process of taking ∇^2, we see that $\nabla^{2p}u_{2p}$ will vanish unless $l=m$, but that

$$\nabla^{2p}\{(a_1x+b_1y+c_1z)^p(a_2x+b_2y+c_2z)^p\} = 2^p(p\,!)^2(a_1a_2+b_1b_2+c_1c_2)^p \ldots\ldots\ldots\ldots (21)$$

Hence, if l, m are different,

$$\iint (a_1x+b_1y+c_1z)^l\,(a_2x+b_2y+c_2z)^m dS_a = 0\,; \ldots\ldots\ldots (22)$$

but

$$\iint (a_1x+b_1y+c_1z)^p(a_2x+b_2y+c_2z)^p dS_a = 4\pi a^{2p+2}\frac{2^p(p\,!)^2}{(2p+1)\,!}(a_1a_2+b_1b_2+c_1c_2)^p. \ldots\ldots\ldots (23)$$

5. These results may be generalised so as to give expressions for the surface integral of any two spherical harmonics. For if u_l is a harmonic of degree l, we may write symbolically

$$u_l(x,\,y,\,z) = \frac{1}{l\,!}\left(x\frac{\partial}{\partial a_1}+y\frac{\partial}{\partial b_1}+z\frac{\partial}{\partial c_1}\right)^l . u_l(a_1,\,b_1,\,c_1) \ldots\ldots (24)$$

where $$\left\{\left(\frac{\partial}{\partial a_1}\right)^2+\left(\frac{\partial}{\partial b_1}\right)^2+\left(\frac{\partial}{\partial c_1}\right)^2\right\} . u_l(a_1,\,b_1,\,c_1) = 0. \ldots\ldots\ldots\ldots (25)$$

Applying the processes of Art. 4 to any harmonics u_l, u_m, u_p, in their symbolical forms, we find

$$\iint u_l u_m dS_a = 0, \text{ if } l \neq m, \ldots\ldots\ldots\ldots (26)$$

but

$$\iint u_p v_p\, dS_a = 4\pi a^{2p+2}\frac{2^p}{(2p+1)\,!}\left(\frac{\partial}{\partial a_1}\frac{\partial}{\partial a_2}+\frac{\partial}{\partial b_1}\frac{\partial}{\partial b_2}+\frac{\partial}{\partial c_1}\frac{\partial}{\partial c_2}\right)^p . u_p(a_1,\,b_1,\,c_1)\,v_p(a_2,\,b_2,\,c_2) \ldots\ldots (27)$$

6. Results analogous to those of the last two articles may be obtained for the integral of the product of three harmonics.

Let
$$\left.\begin{aligned} u_l &= (a_1x + b_1y + c_1z)^l, \\ u_m &= (a_2x + b_2y + c_2z)^m, \\ u_n &= (a_3x + b_3y + c_3z)^n, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (28)$$

where
$$\left.\begin{aligned} a_1^2 + b_1^2 + c_1^2 &= 0, \\ a_2^2 + b_2^2 + c_2^2 &= 0, \\ a_3^2 + b_3^2 + c_3^2 &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

If $l+m+n$ is odd, the integral of the product of the three harmonics obviously vanishes.

Suppose then that $l+m+n$ is even and equal to $2p$.

In order to find $\nabla^{2p}(u_l u_m u_n)$ consider

$$u = \{\lambda(a_1x + b_1y + c_1z) + \mu(a_2x + b_2y + c_2z) + \nu(a_3x + b_3y + c_3z)\}^{2p}$$
$$= \{(\lambda a_1 + \mu a_2 + \nu a_3)x + (\lambda b_1 + \mu b_2 + \nu b_3)y + (\lambda c_1 + \mu c_2 + \nu c_3)z\}^{2p}.$$

We have

$$\nabla^{2p}u = (2p)\,!\{(\lambda a_1 + \mu a_2 + \nu a_3)^2 + (\lambda b_1 + \mu b_2 + \nu b_3)^2 + (\lambda c_1 + \mu c_2 + \nu c_3)^2\}^p$$
$$= (2p)\,!\,2^p\{\mu\nu(a_2a_3 + b_2b_3 + c_2c_3) + \nu\lambda(a_3a_1 + b_3b_1 + c_3c_1)$$
$$+ \lambda\mu(a_1a_2 + b_1b_2 + c_1c_2)\}^p, \quad \ldots(30)$$

on account of (29).

Now equate the coefficients of $\lambda^l\mu^m\nu^n$ on the two sides of (30). On the right side the coefficient will come from a single term involving, say, $(\mu\nu)^\alpha(\nu\lambda)^\beta(\lambda\mu)^\gamma$,

where
$$\beta + \gamma = l, \quad \gamma + \alpha = m, \quad \alpha + \beta = n,$$

so that

$$\alpha = \frac{m+n-l}{2}, \quad \beta = \frac{n+l-m}{2}, \quad \lambda = \frac{l+m-n}{2}.$$

Hence if any one of l, m, n is greater than the sum of the other two, then

$$\nabla^{2p}(u_l u_m u_n) = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

but if any two of l, m, n are together not less than the third, then

$$\frac{(2p)\,!}{l\,!\,m\,!\,n\,!}\nabla^{2p}(u_l u_m u_n) = (2p)!\,2^p\frac{p\,!}{\alpha\,!\,\beta\,!\,\gamma\,!}\,.$$

$$(a_2a_3 + b_2b_3 + c_2c_3)^\alpha(a_3a_1 + b_3b_1 + c_3c_1)^\beta(a_1a_2 + b_1b_2 + c_1c_2)^\gamma \ldots\ldots \quad (32)$$

Thus, by (19)

$$\iint u_l u_m u_n \, dS_a = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33)$$

if any one of l, m, n is greater than the sum of the other two; otherwise

$$\iint u_l u_m u_n dS_a$$

$$= 4\pi a^{l+m+n+2} \frac{2^{\frac{l+m+n}{2}} \left(\frac{l+m+n}{2}\right)! \, l! \, m! \, n!}{(l+m+n+1)! \left(\frac{m+n-l}{2}\right)! \left(\frac{n+l-m}{2}\right)! \left(\frac{l+m-n}{2}\right)!}.$$

$$(a_2a_3 + b_2b_3 + c_2c_3)^{\frac{m+n-l}{2}} (a_3a_1 + b_3b_1 + c_3c_1)^{\frac{n+l-m}{2}} (a_1a_2 + b_1b_2 + c_1c_2)^{\frac{l+m-n}{2}}. \quad (34)$$

Verify that this reduces to (23) when $l = p$, $m = p$, $n = 0$.

The generalised results for any three spherical harmonics may now be written down at once as at (26), (27).

Some Graphical Methods for Determining Centres of Gravity.

By Dr R. F. Muirhead.

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I. Polygonal Plane Areas (First Method).

This method is associated with the well-known construction of a triangle equal in area to a given polygon.

In Fig. 1 let $ABCDD_1C_1B_1A$ be the polygonal circuit enclosing the area, and let DD_1 be any side not adjacent to A.

Let BC' $\|$ to AC meet CD in C'.
,, $C'D'$,, ,, AD ,, DD_1 ,, D'.
,, B_1C_1' ,, ,, AC_1 ,, C_1D_1 ,, C_1'.
,, $C_1'D_1'$,, ,, AD_1 ,, D_1D ,, D_1'.
,, Cc ,, ,, BD ,, BC' ,, c.
,, Dd ,, ,, $D_1'c$,, $D'c$,, d.
,, C_1c_1 ,, ,, B_1D_1 ,, B_1C_1' ,, c_1.
,, D_1d_1 ,, ,, $D'c_1$,, $D_1'c_1$,, d'.

Let O be the mid point of $D'D_1'$, and let AO be produced to A′ so that $OA' = AO$.

Complete the parallelograms $dD'A'd'$, $d_1D_1'd'd_1'$.

Join Ad_1' and from it cut off AG equal to $\frac{1}{3}Ad_1'$.

Then G is the centre of gravity of the given polygonal area.

To explain the construction we note that the triangle $AD'D_1'$ is equal in area to the given polygon. The rest of the construction depends on the principle that if a part of a body have its centre of gravity displaced by a step s, the corresponding displacement of the centre of gravity of the whole will be $\equiv \alpha s$ where α is the ratio of the mass of the part to the mass of the whole.

N.B.—Here, and in what follows, we use the symbol $\equiv$ to indicate equality of length combined with sameness of direction and sense.

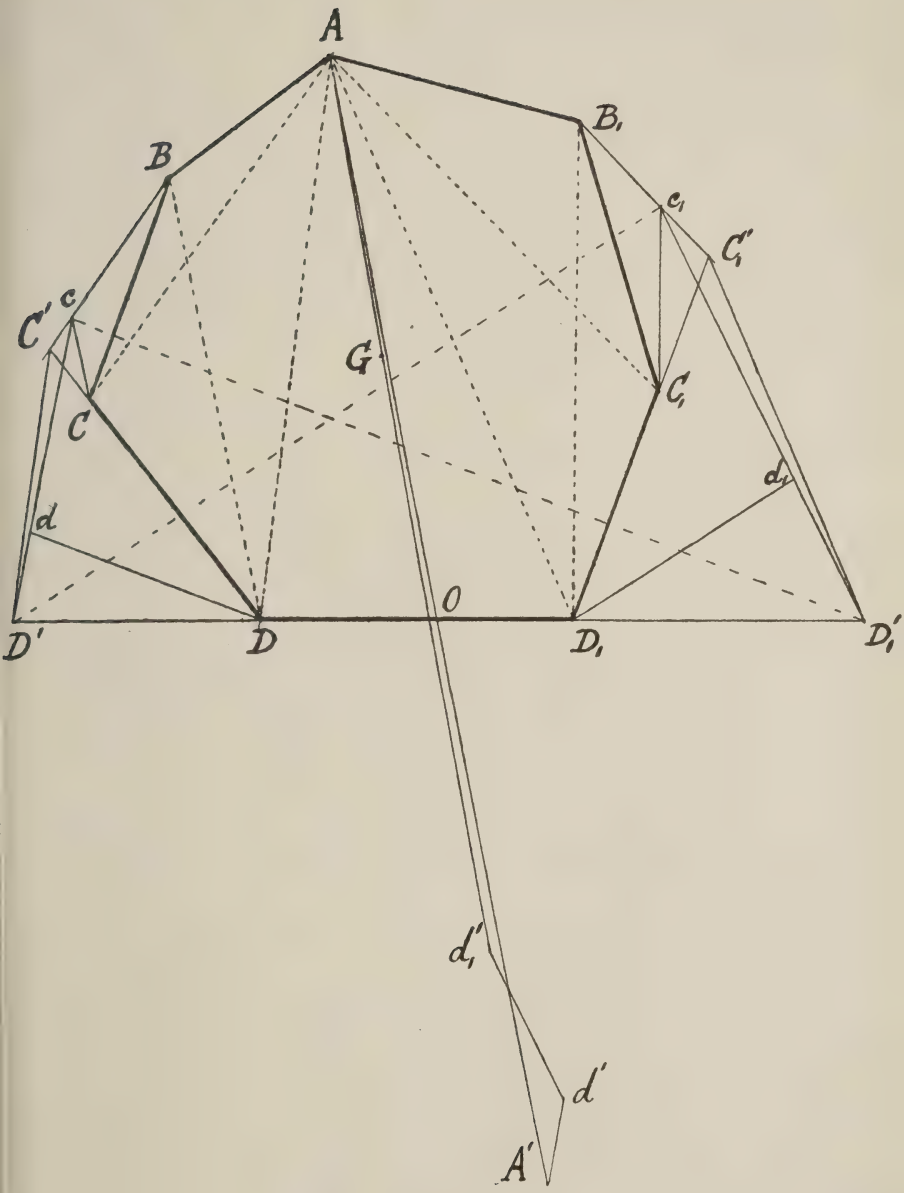

Fig. 1.

Now if the triangular area ABC is shifted to AC'C, its centre of gravity is displaced through a step which is $\equiv \frac{1}{3}BC'$. It follows that the centre of gravity of the area which was originally ABCDA and becomes changed to AC'DA is displaced through a step which $\equiv \frac{1}{3}cC'$, since

$$cC' : BC' = CC' : DC' = \text{area } ABC : \text{area } ABCDA.$$

Again if the triangle AC'D is replaced by the triangle AD'D, its centre of gravity gets a further displacement which $\equiv \frac{1}{3}C'D'$ so that in replacing ABCDA by AD'DA the displacement of its centre of gravity is compounded of $\frac{1}{3}$ of CC' and $\frac{1}{3}$ of C'D', *i.e.* is $\equiv \frac{1}{3}cD'$, and the consequent displacement of the centre of gravity of the whole polygonal area is $\equiv \frac{1}{3}dD'$, since

$$dD' : cD' = DD' : D_1'D' = \text{area } ABCDA : \text{area } ABCDD_1C_1B_1A.$$

Similarly in replacing the area $AB_1C_1D_1A$ by the area $AD_1'D_1A$, the centre of gravity of the whole has been displaced through $\frac{1}{3}$ of the step d_1D_1', so that we can reach the centre of gravity of the original area by constructing that of the triangle $AD'D_1'$ and making a step from that, which is $\frac{1}{3}$ of the step compounded of $D'd$ and $D_1'd_1$.

It follows that G is the centre of gravity of the given area.

It may be remarked that this method of construction is applicable to any plane area, whether forming a single polygon, or several, if it is bounded by straight lines, and whether the polygons are convex or not. It even applies to areas bounded by autotomic rectilinear circuits, if we keep to the usual conventions as to the sign and "multiplicity" of the various compartments, as explained by De Morgan ("Extension of the Word Area," *Camb. and Dub. Math. Journal*, Vol. 5 (1850), p 139), and earlier by A. L. F. Meister ("De Genesi et Indole Figurarum Planarum," Nov. Com. Gött, Tom. I. (1769–70), p. 144).

In the case of an area bounded by several separate polygonal circuits, it is only necessary to introduce double straight lines leading from a point in the perimeter of one disconnected part to a point in the perimeter of another, and back, and to continue this process till all the separate parts are joined up so as to form an area bounded by a single rectilinear circuit which includes each of these double lines, traversed once in one sense and once in the opposite sense.

For example, in Fig. 2 the two areas ABCD and EFG being given, the single circuit ABCDEFGEDA bounds the whole area, the connecting line DE being traversed in both directions.

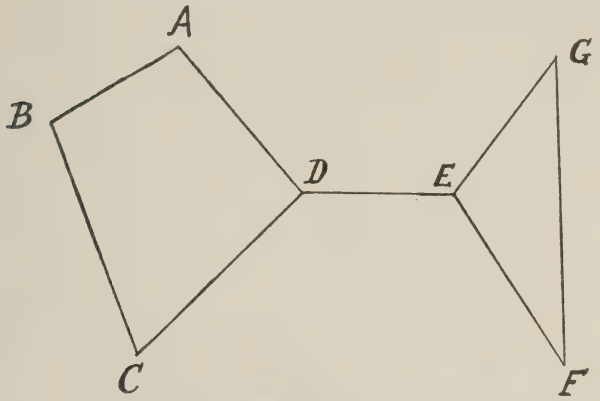

Fig. 2.

The method of construction as applied to the general case of the area enclosed by a single circuit $AB_1B_2 \ldots B_mC_nC_{n-1}C_{n-2} \ldots C_2C_1A$ may be stated thus:—

Let B_1B_2 ∥ to AB_2 meet B_2B_3 in B_2'.

.....

„ $B_r'B'_{r+1}$ „ „ AB_{r+1} „ $B_{r+1}B_{r+2}$ „ B'_{r+1}.

.....

„ $B'_{m-1}B_m'$ „ „ AB_m „ B_mC_n „ B_m'

followed by a similar construction, interchanging C's and B's, m and n. Then

Let B_2b_2 ∥ to B_1B_3 meet B_1B_2' in b_2.

.....

„ B_rb_r „ „ $b_{r-1}B_{r+1}$ „ $b_{r-1}B_r'$ „ b_r.

.....

„ B_mb_m „ „ $b_{m-1}C_n'$ „ $b_{m-1}B_m'$ „ b_m

with a similar construction, interchanging C's and B's, m and n.

Let $B_m' O \equiv OC_n'$, $AO \equiv OA'$, $A'b_m' \equiv B_m' b_m$, $b_m' c_n' \equiv C_n' c_n$, $AG \equiv \frac{1}{3}AC_n'$.

Then G is the required centre of gravity.

II. Centre of Gravity of a System of Particles.

Let n particles $m_1, m_2, \ldots m_n$ be situated at points $A_1, A_2, \ldots A_n$; and let A_{12} be the centre of gravity of m_1 and m_2, A_{23} that of m_2 and m_3, and, generally, let A_{rs} be that of m_r and m_s.

Let the straight lines A_1A_2 and $A_{12}A_3$ intersect in A_{123}.

,, ,, ,, ,, $A_{12}A_{34}$,, $A_{123}A_4$,, ,, A_{1234}.

.....

,, ,, ,, ,, $A_{12\ldots r-1}A_{r,\,r-1}$,, $A_{12\ldots r}A_{r+1}$,, ,, $A_{12\ldots r,\,r+1}$

.....

,, ,, ,, ,, $A_{12\ldots n-2}A_{n-1,\,n}$,, $A_{12\ldots,n-1}A_n$,, ,, $A_{12\ldots n-1,\,n}$.

Then $A_{12\ldots n-1,\,n}$ is the centre of gravity of the system. The reason is obvious.

This construction fails when the particles are all in a straight line. In this case we may introduce a new particle m_0 at A_0, a point not in line with the rest, and having constructed the point $A_{012\ldots n}$ by the above method, draw $A_0A_{012\ldots n}$ to meet the line of the given system in G, which will be the centre of gravity of the system $m_1, m_2, \ldots m_n$.

III. Polygonal Plane Areas (Second Method.)

It will suffice to illustrate this method, which is based on II., by a single example. (See Figure 3).

Let $AB_1B_2B_3B_4B_5A$ be a plane polygon, and let $\beta_1, \beta_2, \beta_3, \beta_4$ be the mid points of B_1B_2, B_2B_3, B_3B_4 and B_4B_5 respectively.

Let $\beta_1\beta_2$ meet AB_2 in C_2 and let $\beta_1\beta_{12} \equiv C_2\beta_2$.

,, $\beta_2\beta_3$,, AB_3 ,, C_3 ,, ,, $\beta_2\beta_{23} \equiv C_3\beta_3$.

,, $\beta_3\beta_4$,, AB_4 ,, C_4 ,, ,, $\beta_3\beta_{34} \equiv C_4\beta_4$.

Let $\beta_1\beta_{23}$ meet $\beta_{12}\beta_3$ in β_{123}.

,, $\beta_{12}\beta_{34}$,, $\beta_{123}\beta_4$,, β_{1334}.

Let $AG \equiv \frac{2}{3}A\beta_{1234}$. Then G is the centroid.

To prove this result, we note the steps from A to the centres of gravity of the triangles AB_1B_2 and AB_2B_3 are respectively $\frac{2}{3}$ of $A\beta_1$ and $\frac{2}{3}$ of $A\beta_2$, so that the step from A to the centre of gravity of the figure $AB_1B_2B_3A$ will be $\frac{2}{3}$ $A\beta_{12}$. Similarly the steps from A to the centres of gravity of the figures $AB_2B_3B_4A$ and $AB_3B_4B_5A$ will be respectively $\frac{2}{3}$ of $A\beta_{23}$ and $\frac{2}{3}$ of $A\beta_{34}$. Hence by II. G is the centroid of the area of the polygon $AB_1B_2B_3B_4B_5A$.

This construction can be modified in the following manner:— Take $A\gamma_1 \equiv \frac{2}{3}A\beta_1$, $A\gamma_2 \equiv \frac{2}{3}A\beta_2$, etc., and in the rest of the construction substitute γ_1, $\gamma_2 \ldots \gamma_{12}$, etc., for β_1, β_2, ... β_{12}, etc. Then γ_{1234} is the required centre of gravity. With this modification it is unnecessary, in dissecting the polygon into triangles, to draw all the dissecting lines from the same vertex A. Any dissection of the polygon into triangles will serve, provided the triangles form a succession such that every pair of adjacent triangles in the succession have one side in common.

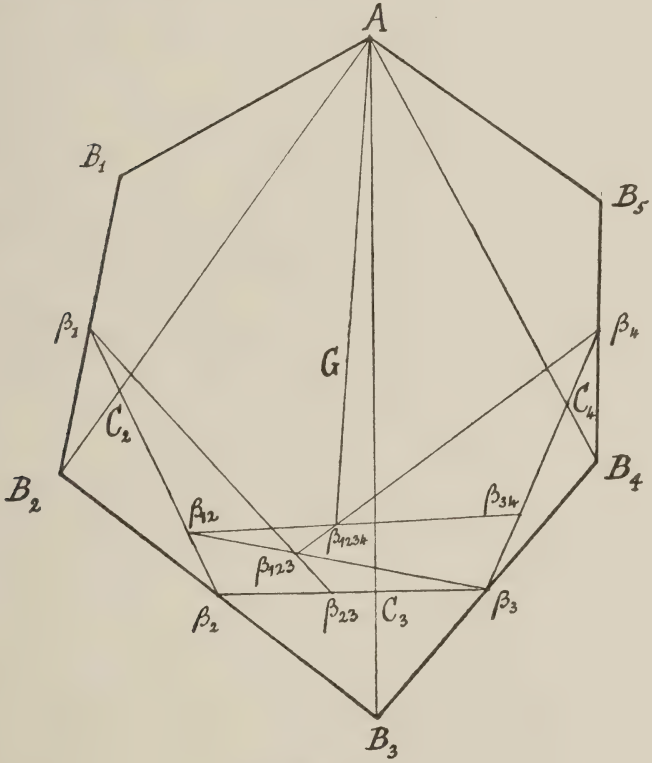

Fig. 3.

This method is, geometrographically considered, simpler than that of Section I., but it is to be noted that if set squares are used there is no need to draw the lines shown dotted in Fig. 1.

IV. Additional Remarks.

The method of II. is not confined to plane systems. And it is to be noted that we can apply this method to construct the centre of gravity of any system of bodies once we have obtained their centres of gravity and the centres of gravity of adjacent pairs when they are arranged in a succession or series, in any order.

For example to find the centre of gravity of a polyhedron, first suppose it divided by plane interfaces into tetrahedrons, and let these have an order in succession such that any two successive tetrahedrons have one face in common. It is then easy to construct the centre of gravity of each tetrahedron and of each pair of adjacent tetrahedrons in the succession. The problem is then reduced to that of II.

Section II. suggests a rather extended application of statical reasoning to prove geometrical propositions, analogous to the statical proof of the concurrence of the medians of a triangle.

If we have a set of points (1), (2), (3)...(n) (1, 2), (1,3)...(1, 2, 3), etc., which are situated so that the following sets of three are each collinear triads, viz.,

(1), (1, 2), (2) : (2), (2, 3), (3) ; ..... ($n-1$), ($n-1$, n), (n) ;

(1), (1, 2, 3) (2, 3) ; (1, 2) (1, 2, 3), (3) ; etc.

.....

r.....(1, ...$r-1$), (1,...r), (r) ; (1,...$r-2$), (1,...r) ($r-1$, r) ;.....

[where (1...r) is a contraction for (1, 2, 3...r)].

Then a similar set of collinearities will hold good for the system of points when the fundamental set (1), (2)...(n) are taken in any other order of succession, and the points (a, b, c ... k), (p, q, ... t) and (a, b, c ... k, p, q...t) will be collinear when a, b, c...k, p, q,...t are any of the numbers 1, 2, ... n.

Thus when $n=3$ we can choose arbitrarily the point (1, 2, 3), and the remaining points (1, 2), (1, 3), (2, 3) are thereby determined by intersections of pairs of lines, or we can choose, say, the points (1, 2), (1, 3) arbitrarily on the lines joining (1) and (2), (1) and (3), after which the two remaining points (2, 3) and (1, 2, 3) are determined.

When $n=4$ we may choose arbitrarily on certain lines the three points (1, 2), (2, 3), (3, 4), after which the remaining derived points (1, 3), (1, 4), (2, 4), (1, 2, 3), (2, 3, 4), (3, 4, 1), (1, 2, 4), (1, 2, 3, 4)

are determined by intersections of pairs of lines, giving altogether 19 sets of 3 collinear points: but in this case there are 6 triads of points which are collinear though the lines have not been drawn. Thus there are six collinearity theorems for such a set of points.

In the general case where n has any positive integral value, the total number of the principal and derived points is $2^n - 1$. Amongst these are $\frac{1}{2}(3^n + 1) - 2^n$ triads of collinear points. Of these collinearities, $2^{n+1} - 3n - 1$ are given in the construction of the derived points, leaving $\frac{1}{2}(3^n + 3) - 3 \cdot 2^n + 3n$ as the number of triads, the assertion of the collinearity of each of which is a *theorem*. Thus for $n = 3$ there are no theorems of this sort; for $n = 4$ there are six theorems; for $n = 5$ there are forty-two theorems, and so on.

The dually correlative theory, in which concurrent lines take the place of collinear points, and *vice versa*, will obviously be true also. It is associated with a method of constructing the line of action of the resultant of any number of coplanar forces acting along lines (1), (2), (3), ...(n), when the lines of action (1, 2), (2, 3) ..($n - 1$, n) of the resultants of successive pairs have been determined. The line (1, 2, 3) is got by joining the intersections of (1, 2) with (3) and of (1) with (2, 3); then the line (1, 2, 3, 4) by joining the intersections of (1, 2, 3) with (4) and of (1, 2) with (3, 4), and so on, until we get the line (1, 2, 3...n), which will be the line of action of the resultant.

This theory and method, however, is not applicable in general to the case of non-coplanar forces, but only if the lines whose mutual intersections should determine points in the figure do really intersect one another.

The construction of the line (1, 2, 3...n) also fails in the case of coplanar forces if the forces are all parallel. But in such a case we can introduce a new force (0) and construct the line (0, 1, 2, ... n); then the line of action (1, 2, ... n) of the original set will be parallel to them and will pass through the intersection of (0, 1, 2 ... n) with the line (0). By a similar expedient we can evade the difficulty arising from the parallelism of any two lines whose intersection is required.

Theorems analogous to Newton's Theorems on Symmetric Functions, with Elementary Proofs.

By Dr R. F. Muirhead.

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In his article in No. 15 of *Mathematical Notes*, Dr Dougall has shown how by introducing series with n terms and a remainder, instead of infinite series, some well-known proofs of Newton's Theorems and others can be rendered more elementary in character. Once the idea has been suggested, it is of course easy to find further applications for it.

In what follows I propose to give some formulae which are closely analogous to Newton's though not so well known, and to furnish them with proofs in which Dr Dougall's idea is further applied.

Notation :—

I use $s_1, s_2, \ldots s_r \ldots$ for the sums of the 1st, 2nd, $\ldots r^{\text{th}}$ powers of n letters $\alpha, \beta, \gamma \ldots \lambda$; and $p_1, p_2, \ldots p_r \ldots p_n$ for the elementary symmetric functions of these letters; and $H_1, H_2, \ldots H_r \ldots$ for the complete sums of homogenous products of the letters, of degrees 1, 2, $\ldots r, \ldots$ Thus p_r stands for $\Sigma(\alpha\beta \ldots$ to r factors) and denotes what is denoted by $(-1)^r p_r$ in Dr Dougall's article.

The following Lemmas will be proved :—

(i) $$\Pi(1-\alpha x).\{1+H_1x+H_2x^2+\ldots+H_mx^m\}=1+x^{m+1}R_m$$

(ii) $$\Sigma\frac{\Pi(1-\alpha x)}{1-\alpha x}=n-(n-1)p_1x+(n-2)p_2x^2\ldots+p_{n-1}(-x)^{n-1}$$

(iii) $$\Sigma\{(1+2\alpha x+3\alpha^2x^2+..+(m+1)\alpha^mx^m)(1+\beta x+\beta^2x^2+..+\beta^mx^m)\ldots(1+\lambda x+\ldots+\lambda^mx^m)\}$$
$$=n+(n+1)H_1x+(n+2)H_2x^2+\ldots+(n+m)H_mx^m+x^{m+1}R_m$$

(iv) $$\Pi(1-\alpha x).\Sigma\{(1+2\alpha x+3\alpha^2x^2+\ldots+(m+1)\alpha^mx^m)(1+\beta x+\ldots+\beta^mx^m)\ldots(1+\lambda x+\ldots+\lambda^mx^m)\}$$
$$=n+s_1x+s_2x^2+\ldots+s_mx^m+x^{m+1}R_m,$$

where in each case the product-symbol Π and the summation symbol Σ extend over n values of the typical factor or term, in which each of the n letters α, $\beta \ldots \lambda$ is in succession substituted for α. R_m denotes in each case a polynomial in x with a finite number of terms; and m is any positive integer.

The formulae, analogous to those of Newton, and deduced from these Lemmas are as follows:—

I. $H_1 - p_1 = 0$

$H_2 - H_1 p_1 + p_2 = 0$

.....

$H_r - H_{r-1} p_1 + H_{r-2} p_2 \ldots - (-1)^r H_1 p_{r-1} + (-1)^r p_r = 0.$

II. $H_1 p_1 - 2p_2 = s_2$

$H_2 p_1 - 2H_1 p_2 + 3p_3 = s_3$

.....

$H_{r-1} p_1 - 2H_{r-2} p_2 + \ldots + (-1)^r (r-1) H_1 p_{r-1} - (-1)^r r p_r = s_r.$

III. $s_r + s_{r-1} H_1 + s_{r-2} H_2 + \ldots + s_1 H_{r-1} = r H_r$

and for comparison, we may write down Newton's formulae with the same notation, viz.:—

IV. $s_1 - p_1 = 0$

$s_2 - s_1 p_1 + 2p_2 = 0$

.....

$s_r - s_{r-1} p_1 + s_{r-2} p_2 - \ldots - (-1)^r s_1 p_{r-1} + (-1)^r p_r = 0.$

In all these formulae it is to be noted that r may have any positive integral value, but when $r > n$ we have

$$p_{n+1} = p_{n+2} = \ldots = p_r = 0.$$

To prove Lemma (i), note that

$$(1 - \alpha x)(1 + \alpha x + \alpha^2 x^2 \ldots + \alpha^m x^m) = 1 - \alpha^{m+1} x^{m+1}$$

$$\therefore \quad \Pi(1 - \alpha x) \,.\, \Pi(1 + \alpha x + \alpha^2 x^2 + \ldots + \alpha^m x^m) = \Pi(1 - \alpha^{m+1} x^{m+1}).$$

Now

$$\Pi(1 + \alpha x + \ldots + \alpha^m x^m) = 1 + H_1 x + H_2 x^2 + \ldots + H_m x^m + x^{m+1} R_m'$$

where $R_m' \equiv$ a polynomial in x.

And $\Pi(1 - \alpha^{m+1} x^{m+1}) = 1 + R_m'' x^{m+1}$, where $R_m'' \equiv$ a polynomial in x.

Hence

$$\Pi(1-\alpha x).\{1+H_1x+H_2x^2+\ldots+H_mx^m\} = 1+x^{m+1}(R_m'-\Pi(1-\alpha x)R_m'') = 1+x^{m+1}R_m$$

where R is a finite polynomial in x.

Proof of Lemma (ii):

We have $$\frac{\Pi(1-\alpha x)}{1-\alpha x} \equiv (1-\beta x)(1-\gamma x)\ldots(1-\lambda x) = 1-a_1x+a_2x^2-\ldots+a_{n-1}(-x)^{n-1}$$

where $a_r \equiv$ the elementary symmetric function of degree r of the $n-1$ letters $\beta, \gamma, \ldots\lambda$.

Similarly $$\frac{\Pi(1-\alpha x)}{(1-\beta x)} = 1-b_1x+b_2x^2\ldots+b_{n-1}(-x)^{n-1}$$

where $b_r \equiv$ the value of a_r with α and β interchanged, and so on.

Hence $$\Sigma\frac{\Pi(1-\alpha x)}{1-\alpha x} = n-x\Sigma a_1+x^2\Sigma a_2\ldots+(-x)^{m-1}\Sigma a_{n-1}$$

the summation extending over n terms, one of the letters $\alpha, \beta\ldots\lambda$ being omitted each time.

It is obvious that $\Sigma a_r = (n-r)p_r$, since any particular combination occurs as often as there are letters not involved in it, viz., $n-r$ times.

Hence

$$\Sigma\frac{\Pi(1-\alpha x)}{1-\alpha x} = n-(n-1)\,p_1x+(n-2)p_2x^2-\ldots+p_{n-1}(-x)^{n-1}.$$

To prove Lemma (iii), consider the expression:

$$\Sigma\{(1+2\alpha x+3\alpha^2x^2+\ldots+\overline{m+1}\alpha^mx^m)(1+\beta x+\ldots+\beta^mx^m) (1+\gamma x+\ldots+\gamma^mx^m)\ldots(1+\lambda x+\ldots+\lambda^mx^m)\}$$

where the summation extends over n terms, in each of which *one* of the letters $\alpha, \beta, \ldots\lambda$ occurs in a factor similar to that involving α in the typical term attached to the Σ. If this expression be expanded, we get a set of terms of the type $C\alpha^a\beta^b\ldots\lambda^lx^{a+b\ldots+l}$, where C is a numerical factor and the indices $a, b, \ldots l$ are positive integers or zeros. The terms having the same values of $a, b, \ldots l$ are n in number, and each has one of the values $a+1, b+1, \ldots l+1$ as numerical coefficient. Hence the sum of such terms is

$$\{(a+1)+(b+1)+\ldots+(l+1)\}\alpha^a\beta^b\ldots\lambda^lx^{a+b\ldots+l}$$

so that the numerical coefficient $= n+\Sigma a = n+r$ where r is the index of the power of x.

Thus the coefficient of x^r in the expansion is

$$(n+r)\Sigma(\alpha^a\beta^b\gamma^c\ldots\lambda^l),$$

where $a, b, c\ldots$ take all positive integral or zero values that make $\Sigma a = r$, so that $\Sigma(\alpha^a\beta^b\ldots\lambda^l) = \mathrm{H}_r$, provided $m \not< r$.

Hence we have

$$\Sigma\{(1+2\alpha x+3\alpha^2x^2+\ldots+(m+1)\alpha^m x^m)(1+\beta x+\ldots+\beta^m x^m)\ldots (1+\lambda x+\ldots+\lambda^m x^m)\}$$
$$= n+(n+1)\mathrm{H}_1x+(n+2)\mathrm{H}_2x^2+\ldots+(n+r)\mathrm{H}_r+\ldots+(n+m)\mathrm{H}_m x^m$$
$$+\text{a finite polynomial in } x \text{ having no power of } x \text{ lower than } x^{m+1}.$$

To prove Lemma (iv):

$$(1+2\alpha x+3\alpha^2x^2+\ldots+(m+1)\alpha^m x^m)(1-\alpha x) = 1+\alpha x+\alpha^2x^2\ldots+\alpha^m x^m-\overline{m+1}\alpha^{m+1}x^{m+1}$$

$$(1+\beta x+\beta^2x^2+\ldots+\beta^m x^m)(1-\beta x) = 1-\beta^{m+1}x^{m+1}, \text{ etc.}$$

Hence

$$\{1+2\alpha x+3\alpha^2x^2+\ldots+(m+1)\alpha^m x^m\}(1+\beta x+\beta^2x^2+\ldots+\beta^m x^m)\ldots(1+\lambda x\ldots+\lambda^m x^m)\Pi(1-\alpha x)$$
$$=\{1+\alpha x+\alpha^2x^2+\ldots+\alpha^m x^m-(m+1)\alpha^{m+1}x^{m+1}\}\times(1-\beta^{m+1}x^{m+1})\ldots(1-\lambda^{m+1}x^{m+1})$$
$$=1+\alpha x+\alpha^2x^2\ldots+\alpha^m x^m+x^{m+1}\times(\text{a finite polynomial in } x).$$

Summing, over n equations of this type, we have

$$\Pi(1-\alpha x).\Sigma\{(1+2\alpha x+3\alpha^2x^2\ldots+(m+1)\alpha^m x^m)(1+\beta x+\beta^2x^2\ldots+\beta^m x^m)\ldots(1+\lambda x\ldots+\lambda^m x^m)\}$$
$$=n+s_1x+s_2x^2\ldots+s_m x^m+x^{m+1}\times(\text{a finite polynomial in } x).$$

To prove Formulae I., we have by Lemma (i)

$$\{1-p_1x+p_2x^2\ldots+p_n(-x)^n\}\{1+\mathrm{H}_1x+\mathrm{H}_2x^2+\ldots+\mathrm{H}_m x^m\} = 1+x^{m+1}\mathrm{R}_m.$$

Equating coefficients of $x, x^2, \ldots x^r$, we get the formulae as stated above.

To prove Formulae II, we have by Lemmas (iii) and (iv)

$$\{1-p_1x+p_2x^2\ldots+p_n(-x)^n\}\{n+(n+1)\mathrm{H}_1x+\ldots+(n+2)\mathrm{H}_2x^2+\ldots+(n+m)\mathrm{H}_m x^m\}$$
$$=n+s_1x+s_2x^2+\ldots+s_m x^m+\text{a finite polynomial in } x \text{ containing only powers higher than } x^m.$$

Hence, equating coefficients of x^r,

$$(n+r)\mathrm{H}_r - p_1(n+r-1)\mathrm{H}_{r-1} + p_2(n+r-2)\mathrm{H}_{r-2} - \ldots$$
$$\ldots + (-1)^{r-1}p_{r-1}(n+1)\mathrm{H}_1 + (-1)^r np_r + s_r.$$

But by I. $(n+r)(\mathrm{H}_r - p_1\mathrm{H}_{r-1} + p_2\mathrm{H}_{r-2}\ldots + (-1)^r p_r) = 0.$

Subtracting, we get

$$\mathrm{H}_{r-1}p_1 - 2\mathrm{H}_{r-2}p_2 + 3\mathrm{H}_{r-3}p_3\ldots + (-1)^r(r-1)\mathrm{H}_1 p_{r-1} - (-1)^r r p_r = s_r$$

which is the type-formula of II.

To prove formulae III, we multiply both sides of the identity in Lemma (iv) by $(1 + \mathrm{H}_1 x + \mathrm{H}_2 x^2 + \ldots + \mathrm{H}_m x^m)$. and making use of Lemmas (i) and (iii) we get

$$(1+x^{m+1}\mathrm{R}_m)\{n + (n+1)\mathrm{H}_1 x + (n+2)\mathrm{H}_2 x^2 + \ldots$$
$$\ldots + (n+m)\mathrm{H}_m x^m + x^{m+1}\mathrm{R}_m'\}$$
$$= (n + s_1 x + s_2 x^2 + \ldots + s_m x^m + x^{m+1}\mathrm{R}_m'')(1 + \mathrm{H}_1 x + \ldots + \mathrm{H}_m x^m).$$

Equating coefficients of x^r we get

$$(n+r)\mathrm{H}_r = n\mathrm{H}_r + s_1\mathrm{H}_{r-1} + s_2\mathrm{H}_{r-2} + \ldots + s_{r-1}\mathrm{H}_1 + s_r$$

which at once reduces to the typical formula in III.

On Certain Formulae connected with the Function $(\cos n\alpha - \cos n\theta)/(\cos\alpha - \cos\theta)$.

By D. G. Taylor, M.A.

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§ 1. This paper deals with certain formulae which, though probably not all new, have not appeared in the text-books. They were suggested to the writer while engaged in discussing the expression for the intensity of the transmitted beam in the *Lummer-Gehrcke* Interference Spectroscope, *viz.*

$$J = \frac{1 - 2p^n \cos n\theta + p^{2n}}{1 - 2p\cos\theta + p^2},$$

where p is the square of the reflexion-coefficient, which in practice falls not far short of unity, $2n$ is the number of internal reflexions, and θ is the phase-difference of successive elementary beams.* If we put $k = log_e p$, we have

$$J = p^{n-1}\frac{chnk - \cos n\theta}{chk - \cos\theta},$$

which at once suggests the function in the title.

The function J has a very high maximum value for $\theta = 0, 2\pi, \ldots$, yielding sharp bright bands with feebly luminous interspaces; and the sharpness of the bands is measured by the value of $d^2J/d\theta^2$ for these values of θ. The most useful formula, obtained by direct differentiation, is

$$\left(\frac{d^2J}{d\theta^2}\right)_{\theta=0} = -2p\frac{(1-p^n)^2 - n^2p^{n-1}(1-p)^2}{(1-p)^4},$$

and the limit of this expression for $p \to 1$ is $-\frac{1}{6}n^2(n^2-1)$.

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* See Dr E. Gehrcke, *Die Anwendung der Interferenzen in der Spectroskopie und Metrologie*, (Braunschweig, 1906), p. 69. I have modified the notation. The subject was brought to my notice by Prof. A. L. Selby.

The following specimen values were obtained by this method : †

n	p	$(d^2J/d\theta^2)_{\theta=0}$
15	·8	535
15	·883	1649
15	·9	2097
15	1·0	8400
20	·883	3070
20	1·0	26600

But it is clear that J could be differentiated or integrated any number of times if it were expressed as a finite Fourier series, in cosines of multiples of θ.

§ 2. The expression

$$\frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

can be written

(i) as a product of $(n-1)$ factors each linear in $\cos\theta$, thus:

$$\frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos\theta} = 2^{n-1}\prod_{r=1}^{n-1}\left\{\cos\theta - \cos\left(\alpha + \frac{2r\pi}{n}\right)\right\},$$

(*cf.* Hobson, *Plane Trig.*, § 188) ;

(ii) as a finite series of powers of $\cos\theta$, up to the $(n-1)^{\text{th}}$; to be obtained by multiplying out the factors above, or from the relation (*ibid.* § 78)

$$2\cos n\theta = (2\cos\theta)^n - \frac{n}{1!}(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4}\ldots ;$$

(iii) in the form we propose to consider, as a linear function of cosines of multiples of θ, up to the $(n-1)^{\text{th}}$.

† $n=15$, $p=·883$ are the actual values in a typical case.

§ 3. *To find the coefficients in the identity*

$$\frac{chnk - \cos n\theta}{chk - \cos\theta} = \tfrac{1}{2}a_0 + \sum_{r=1}^{n-1} a_r \cos r\theta.$$

On multiplying up we have

$$a_{r+1} - 2chk \,.\, a_r + a_{r-1} = 0, \quad (r = 1, 2, \ldots (n-1)), \quad \ldots\ldots\ldots\ldots (2)$$

$$a_0 chk - a_1 = 2chnk, \quad a_{n-1} = 2\,; \quad \ldots\ldots\ldots\ldots (3)$$

from (2), putting $p = e^k$,

$$a_r = \mathrm{A}p^r + \mathrm{B}p^{-r},$$

where A, B are independent of r; and from (3),

$$\mathrm{B} = \frac{2p^n}{p - p^{-1}}, \quad \mathrm{A} = \frac{-2p^{-n}}{p - p^{-1}},$$

whence

$$a_r = \frac{2(p^{n-r} - p^{-n+r})}{p - p^{-1}} = \frac{2sh(n-r)k}{shk},$$

and

$$\frac{chnk - \cos n\theta}{chk - \cos\theta} = \frac{shnk}{shk} + \frac{2}{shk}\sum_{r=1}^{n-1} sh(n-r)k \cos r\theta. \quad \ldots\ldots\ldots\ldots (4)$$

We may note that k is positive or negative according as p is greater or less than unity.

§ 4. From (4) we at once deduce

$$\frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos\theta} = \frac{\sin n\alpha}{\sin\alpha} + \frac{2}{\sin\alpha}\sum_{r=1}^{n-1} \sin(n-r)\alpha \cos r\theta,$$

$$= \frac{\sin n\theta}{\sin\theta} + \frac{2}{\sin\theta}\sum_{r=1}^{n-1} \sin(n-r)\theta \cos r\alpha\,; \quad \ldots\ldots\ldots\ldots (5)$$

$$\frac{chnk - chnu}{chk - chu} = \frac{shnk}{shk} + \frac{2}{shk}\sum_{r=1}^{n-1} sh(n-r)k\, chru,$$

$$= \frac{shnu}{shu} + \frac{2}{shu}\sum_{r=1}^{n-1} sh(n-r)u\, chrk\,; \quad \ldots\ldots\ldots\ldots (6)$$

$$\frac{1 - \cos n\theta}{1 - \cos\theta} = n + 2\sum_{r=1}^{n-1} (n-r)\cos r\theta,$$

$$= \frac{\sin n\theta}{\sin\theta} + \frac{2}{\sin\theta}\sum_{r=1}^{n-1} \sin(n-r)\theta\,; \quad \ldots\ldots\ldots\ldots (7)$$

$$\frac{1-chnu}{1-chu} = n + 2\sum_{r=1}^{n-1}(n-r)chru,$$

$$= \frac{shnu}{shu} + \frac{2}{shu}\sum_{r=1}^{n-1} sh(n-r)u\ ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$\frac{chnk-\cos n\theta}{chk-\cos\theta} = \frac{shnk}{shk} + \frac{2}{shk}\sum_{r=1}^{n-1} sh(n-r)k\cos r\theta,$$

$$= \frac{\sin n\theta}{\sin\theta} + \frac{2}{\sin\theta}\sum_{r=1}^{n-1}\sin(n-r)\theta chrk \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

§ 5. Replacing θ by $\theta+\pi$, and u by $u+i\pi$, we have

$$\frac{\cos n\alpha-(-)^n\cos n\theta}{\cos\alpha+\cos\theta} = \frac{\sin n\alpha}{\sin\alpha} + \frac{2}{\sin\alpha}\sum_{r=1}^{n-1}(-)^r\sin(n-r)\alpha\cos r\theta,$$

$$= -\frac{2}{\sin\theta}\sum_{r=1}^{n-1}(-)^r\cos(n-r)\alpha\sin r\theta - \frac{(-)^n\sin n\theta}{\sin\theta}\ ; \quad \ldots(5')$$

$$\frac{chnk-(-)^n chnu}{chk+chu} = \frac{shnk}{shk} + \frac{2}{shk}\sum_{r=1}^{n-1}(-)^r sh(n-r)kchru,$$

$$= -\frac{2}{shu}\sum_{r=1}^{n-1}(-)^r ch(n-r)kshru - \frac{(-)^n shnu}{shu}\ ; \quad \ldots\ldots\ldots(6')$$

$$\frac{1-(-)^n\cos n\theta}{1+\cos\theta} = n + 2\sum_{r=1}^{n-1}(-)^r(n-r)\cos r\theta,$$

$$= -\frac{2}{\sin\theta}\sum_{r=1}^{n-1}(-)^r\sin r\theta - \frac{(-)^n\sin n\theta}{\sin\theta}\ ; \quad \ldots\ldots\ldots\ldots\ldots(7')$$

$$\frac{1-(-)^n chnu}{1+chu} = n + 2\sum_{r=1}^{n-1}(-)^r(n-r)chru,$$

$$= -\frac{2}{shu}\sum_{r=1}^{n-1}(-)^r shru - \frac{(-)^n shnu}{shu}\ ; \quad \ldots\ldots\ldots\ldots\ldots(8')$$

$$\frac{chnk-(-)^n\cos n\theta}{chk+\cos\theta} = \frac{shnk}{shk} + \frac{2}{shk}\sum_{r=1}^{n-1}(-)^r sh(n-r)k\cos r\theta,$$

$$= -\frac{2}{\sin\theta}\sum_{r=1}^{n-1}(-)^r ch(n-r)k\sin r\theta - \frac{(-)^n\sin n\theta}{\sin\theta}\ldots\ldots(9')$$

Similarly, by substituting $\frac{1}{2}\pi\pm\theta$, $\frac{1}{2}\pi-\alpha$, $\frac{1}{2}i\pi\pm u$, $\frac{1}{2}i\pi-k$ for θ, α, u, k respectively, a large variety of formulae can be deduced, proceeding on the right by alternate sines and cosines of multiples of θ (or alternate hyperbolic sines and cosines of multiples of u), and having for denominators on the left such quantities as

$$\cos\alpha\pm\sin\theta,\ \sin\alpha\pm\sin\theta,\ chk\pm shu,\ shk\pm shu,$$
$$1\pm\sin\theta,\ 1\pm shu,\ chk\pm\sin\theta,\ shk\pm\sin\theta,\ \sin\alpha\pm shu.$$

§ 6. We have (*cf. Hobson*, §74, *ex.* 1).

$$\frac{\sin n\alpha}{\sin\alpha} = 2\cos(n-1)\alpha + 2\cos(n-3)\alpha + \ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

where, however, if n is odd, the final term is *unity*. Also

$$\frac{\cos(2n-1)\alpha}{\cos\alpha} = 2\cos(2n-2)\alpha - 2\cos(2n-4)\alpha\ldots + (-1)^{n-1},$$

$$\frac{(-1)^{n-1} + \cos 2n\alpha}{\cos\alpha} = 2\cos(2n-1)\alpha - 2\cos(2n-3)\alpha\ldots + (-)^{n-1}2\cos\alpha,$$

$$\frac{\sin 2n\alpha}{\cos\alpha} = 2\sin(2n-1)\alpha - 2\sin(2n-3)\alpha\ldots,$$

$$\frac{1-\cos 2n\alpha}{\sin\alpha} = 2\sin(2n-1)\alpha + 2\sin(2n-3)\alpha\ldots,$$

with similar results in hyperbolics. Applying (10) to the coefficients in (5), we obtain an expansion of our function which is linear in the cosines of multiples of α, and also in those of θ; in fact, a *finite double Fourier series*, thus:

$$\frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos} = 4\sum_{r,s}\cos r\alpha\cos s\theta + 2\sum_{t}(\cos t\alpha + \cos t\theta) + c, \quad \ldots\ldots\ldots(11)$$

the summations extending to all positive integral non-zero values of r, s, t for which $n-r-s$, $n-t$ are odd positive integers, and c being zero or unity according as n is even or odd. Thus *e.g.*

$$(\cos 5\alpha - \cos 5\theta)/(\cos\alpha - \cos\theta)$$
$$= 4(\cos 3\alpha\cos\theta + \cos 2\alpha\cos 2\theta + \cos\alpha\cos 3\theta + \cos\alpha\cos\theta)$$
$$+ 2(\cos 4\alpha + \cos 4\theta + \cos 2\alpha + \cos 2\theta) + 1.$$

Similar results will follow for the other formulae of §§ 4, 5. And while the formulae of these articles are suitable for repeated differentiation and integration with regard to one or other of the symbols involved, those of the form (11) will admit of these processes being applied with respect to *both* variables.

§ 7. *Repeated Differentiations.—Expansions.*

By (10),

$$\left\{\frac{d^m}{d\alpha^m}\left(\frac{\sin n\alpha}{\sin\alpha}\right)\right\}_{\alpha=0} = 2n_m \cdot \cos\tfrac{1}{2}m\pi,$$

where
$$n_m = (n-1)^m + (n-3)^m + \ldots,$$

the last term being 1^m or 2^m according as n is even or odd; hence

$$\frac{\sin n\alpha}{\sin\alpha} = n + 2\sum_{m=1}^{\infty}(-)^m n_{2m}\frac{\alpha^{2m}}{(2m)!}.$$

Similarly let

$$n_m' = (n-1)^m - (n-3)^m + \dots;$$

then *when n is even*,

$$\frac{(-)^{\frac{1}{2}n-1} + \cos n\alpha}{\cos\alpha} = c' + 2\sum_{m=1}^{\infty}(-)^m n'_{2m}\frac{\alpha^{2m}}{(2m)!},$$

c' being zero or 2 according as $\frac{1}{2}n$ is even or odd; and *when n is odd*,

$$\frac{\cos n\alpha}{\cos\alpha} = 1 + 2\sum_{m=1}^{\infty}(-)^m n'_{2m}\frac{\alpha^{2m}}{(2m)!}.$$

By (7),

$$\left\{\frac{d^m}{d\theta^m}\left(\frac{1-\cos n\theta}{1-\cos\theta}\right)\right\}_{\theta=0} = 2a_{n,m}\cos\tfrac{1}{2}m\pi,$$

where

$$a_{n,m} = \sum_{r=1}^{n-1}(n-r)r^m;$$

hence

$$\frac{1-\cos n\theta}{1-\cos\theta} = n^2 + 2\sum_{m=1}^{\infty}(-)^m a_{n,2m}\frac{\theta^{2m}}{(2m)!}.$$

By (9),

$$\left\{\frac{d^m}{d\theta^m}\left(\frac{chnk-\cos n\theta}{chk-\cos\theta}\right)\right\}_{\theta=0} = \frac{2\cos\frac{1}{2}m\pi}{shk}\sum_{r=1}^{n-1} r^m sh(n-r)k;$$

let us put

$$k_{n,m} = \sum_{r=1}^{n-1}\frac{r^m sh(n-r)k}{shk} = \sum_{r=1}^{n-1} r^m\frac{p^{n-r}-p^{-n+r}}{p-p^{-1}} \quad (m>0),$$

and

$$k_{n,0} = \frac{chnk-1}{chk-1} = \frac{1}{p^{n-1}}\left(\frac{p^n-1}{p-1}\right)^2,\ p=e^k,$$

then

$$\frac{chnk-\cos n\theta}{chk-\cos\theta} = k_{n,0} + 2\sum_{m=1}^{\infty}(-)^m k_{n,2m}\frac{\theta^{2m}}{(2m)!}.$$

§ 8. *Integration.*

(i) From the formulae of § 6 we have

$$\int_0\frac{\sin n\theta\, d\theta}{\sin\theta} = \frac{2\sin(n-1)\theta}{n-1} + \frac{2\sin(n-3)\theta}{n-3} + \dots + c\theta,$$

c being zero or unity according as n is even or odd;

$$\int_0 \frac{\cos(2n-1)\theta d\theta}{\cos\theta} = \frac{2\sin(2n-2)\theta}{2n-2} - \frac{2\sin(2n-4)\theta}{2n-4} \ldots + (-)^{n-1}\theta,$$

$$\int_0 \frac{(-1)^{n-1} + \cos 2n\theta}{\cos\theta} d\theta = \frac{2\sin(2n-1)\theta}{2n-1} - \frac{2\sin(2n-3)\theta}{2n-3} \ldots + (-)^{n-1} 2\sin\theta$$

$$\int_{\frac{1}{2}\pi} \frac{\sin 2n\theta d\theta}{\cos\theta} = -\frac{2\cos(2n-1)\theta}{2n-1} + \frac{2\cos(2n-3)\theta}{2n-3} - \ldots,$$

$$\int_{\frac{1}{2}\pi} \frac{1-\cos 2n\theta}{\sin\theta} d\theta = -\frac{2\cos(2n-1)\theta}{2n-1} - \frac{2\cos(2n-3)\theta}{2n-3} - \ldots.$$

From these we obtain

$$\int_0^{\pi} \frac{\sin 2n\theta d\theta}{\sin\theta} = 0, \qquad \int_0^{\pi} \frac{\sin(2n-1)\theta d\theta}{\sin\theta} = \pi,$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin 2n\theta d\theta}{\sin\theta} = 2\left(1 - \tfrac{1}{3} + \tfrac{1}{5} \ldots + \frac{(-)^{n-1}}{2n-1}\right), \quad \int_0^{\frac{1}{2}\pi} \frac{\sin(2n-1)\theta d\theta}{\sin\theta} = \tfrac{1}{2}\pi;$$

$$\int_0^{\pi} \frac{\cos(2n-1)\theta d\theta}{\cos\theta} = (-)^{n-1}\pi, \qquad \int_0^{\frac{1}{2}\pi} \frac{\cos(2n-1)\theta d\theta}{\cos\theta} = (-)^{n-1}\frac{\pi}{2},$$

$$\int_0^{\pi} \frac{(-)^{n-1} + \cos 2n\theta}{\cos\theta} d\theta = 0, \quad \int_0^{\frac{1}{2}\pi} \frac{(-)^{n-1} + \cos 2n\theta}{\cos\theta} d\theta$$
$$= (-)^{n-1} 2\left(1 + \tfrac{1}{3} + \ldots + \frac{1}{2n-1}\right);$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin 2n\theta d\theta}{\cos\theta} = \int_{\frac{1}{2}\pi}^{\pi} \frac{\sin 2n\theta d\theta}{\cos\theta} = 2\left(\frac{1}{2n-1} - \frac{1}{2n-3} + \ldots\right),$$

$$\int_0^{\frac{1}{2}\pi} \frac{1-\cos 2n\theta}{\sin\theta} d\theta = \int_{\frac{1}{2}\pi}^{\pi} \frac{1-\cos 2n\theta}{\sin\theta} d\theta = 2\left(1 + \tfrac{1}{3} + \ldots + \frac{1}{2n-1}\right).$$

(ii) From (5), (7), (9) we have

$$\int_0 \frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos\theta} d\theta = \frac{\sin n\alpha}{\sin\alpha}\theta + \sum_{r=1}^{n-1} \frac{2\sin(n-r)\alpha \sin r\theta}{r\sin\alpha},$$

$$\int_0 \frac{1-\cos n\theta}{1-\cos\theta} d\theta = n\theta + 2\sum_{r=1}^{n-1} \frac{n-r}{r} \sin r\theta,$$

$$\int_0 \frac{chnk - \cos n\theta}{chk - \cos\theta} d\theta = \frac{shnk}{shk}\theta + \sum_{r=1}^{n-1} \frac{2sh(n-r)k \sin r\theta}{rshk};$$

hence

$$\int_0^\pi \frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos\theta} d\theta = \frac{\pi \sin n\alpha}{\sin\alpha},$$

$$\int_0^\pi \frac{1 - \cos n\theta}{1 - \cos\theta} d\theta = n\pi,$$

$$\int_0^\pi \frac{chnk - \cos n\theta}{chk - \cos\theta} d\theta = \frac{\pi shnk}{shk}.$$

In the last write $p = e^k$, and obtain

$$\int_0^\pi \frac{1 - 2p^n \cos n\theta + p^{2n}}{1 - 2p\cos\theta + p^2} d\theta = \pi \frac{p^{2n} - 1}{p^2 - 1},$$

where p may have any real value.

(iii) Again from (5), (7), (9), on multiplying throughout by $\cos r\theta$ and integrating from 0 to π, as in finding the coefficients of a Fourier series, we have

$$\int_0^\pi \frac{(\cos n\alpha - \cos n\theta)\cos r\theta d\theta}{\cos\alpha - \cos\theta} = \frac{\pi \sin(n-r)\alpha}{\sin\alpha},$$

$$\int_0^\pi \frac{(1 - \cos n\theta)\cos r\theta d\theta}{1 - \cos\theta} = (n - r)\pi,$$

$$\int_0^\pi \frac{(chnk - \cos n\theta)\cos r\theta d\theta}{chk - \cos\theta} = \frac{\pi sh(n-r)k}{shk};$$

and with p as before,

$$\int_0^\pi \frac{(1 - 2p^n \cos n\theta + p^{2n})\cos r\theta d\theta}{1 - 2p\cos\theta + p^2} = \frac{\pi}{p^r} \frac{p^{2n} - p^{2r}}{p^2 - 1} = \pi p^r \cdot \frac{p^{2(n-r)} - 1}{p^2 - 1}.$$

(iv) The integral

$$\int \frac{\cos n\theta d\theta}{\cos\alpha - \cos\theta}$$

is improper if the range of integration includes α (or more generally $2r\pi \pm \alpha$); but we can write

$$\int_0^\pi \frac{\cos n\theta d\theta}{chk - \cos\theta} = chnk \int_0^\pi \frac{d\theta}{chk - \cos\theta} - \int_0^\pi \frac{chnk - \cos n\theta}{chk - \cos\theta} d\theta.$$

Now the first integral on the right is essentially positive, hence its value is $\pm \frac{\pi chnk}{shk}$, according as k is positive or negative.

Let k be positive, then $p = e^k > 1$, and we have

$$\int_0^\pi \frac{\cos n\theta d\theta}{chk - \cos\theta} = \frac{\pi e^{-nk}}{shk}, \quad \text{.....(12)}$$

whence

$$\int_0^\pi \frac{\cos n\theta d\theta}{1-2p\cos\theta+p^2} = \frac{\pi p^{-n}}{p^2-1}. \ (p>1).$$

If we put $q = p^{-1} = e^{-k} < 1$, we shall have

$$\int_0^\pi \frac{\cos n\theta d\theta}{1-2q\cos\theta+q^2} = \frac{\pi q^n}{1-q^2}. \ (0<q<1).$$

(*cf.* Bromwich, *Infinite Series*, p. 167).

Now writing $chk = \sec\beta$, so that $shk = \tan\beta$, and

$$p = e^k = \sec\beta + \tan\beta = \frac{1+\tan\frac{1}{2}\beta}{1-\tan\frac{1}{2}\beta}, \quad (0<\beta<\tfrac{1}{2}\pi),$$

we have from (12)

$$\int_0^\pi \frac{\cos n\theta d\theta}{1-\cos\beta\cos\theta} = \frac{\pi}{\sin\beta}\left(\frac{1-t}{1+t}\right)^n, \ t = \tan\tfrac{1}{2}\beta. \ \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

The companion formula

$$\int_0^\pi \frac{\cos n\theta d\theta}{1+\cos\beta\cos\theta} = \frac{(-)^n\pi}{\sin\beta}\left(\frac{1-t}{1+t}\right)^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

may be obtained from (13) either by changing θ into $\pi-\theta$, or β into $\pi-\beta$. The formula (13) is a good example of a discontinuous function. The integral becomes improper when β takes the value 0 or π, and therefore, as a function of β, is discontinuous at those values. Thus it is inadmissible to change β into $-\beta$ or into $\pi+\beta$, though it is allowable to change it into $\pi-\beta$.

Another form is obtained by putting $\cos\beta = e$,

$$\int_0^\pi \frac{\cos n\theta d\theta}{1\mp e\cos\theta} = \frac{\pi}{\sqrt{(1-e^2)}}\left\{\frac{\pm e}{1+\sqrt{(1-e^2)}}\right\}^n.$$

(v) Finally from (ii),

$$\int_0^\pi\int_0^\pi \frac{\cos n\alpha - \cos n\theta}{\cos\alpha-\cos\theta} d\alpha d\theta = 0 \text{ or } \pi^2$$

according as n is even or odd, while

$$\int_0^\pi\int_0^\pi \frac{(\cos n\alpha - \cos n\theta)\cos r\alpha\cos s\theta d\alpha d\theta}{\cos\alpha-\cos\theta} = 0 \text{ or } \pi^2$$

according as $n-r-s$ is even or odd.

Concurrency of lines joining vertices of a triangle to opposite vertices of triangles on its sides.

By A. G. Burgess, M.A., F.R.S.E.

(*Read 13th February 1914. Received 10th April 1914*).

(1). Let ABC be the given triangle; A′BC, B′CA, C′BA triangles described externally on its sides, and let the angles of these triangles be $A'BC = \mu_1$, $A'CB = \nu_1$, $B'AC = \lambda_2$, $B'CA = \nu_2$, $C'AB = \lambda_3$, $C'BA = \mu_3$ (Fig. 1).

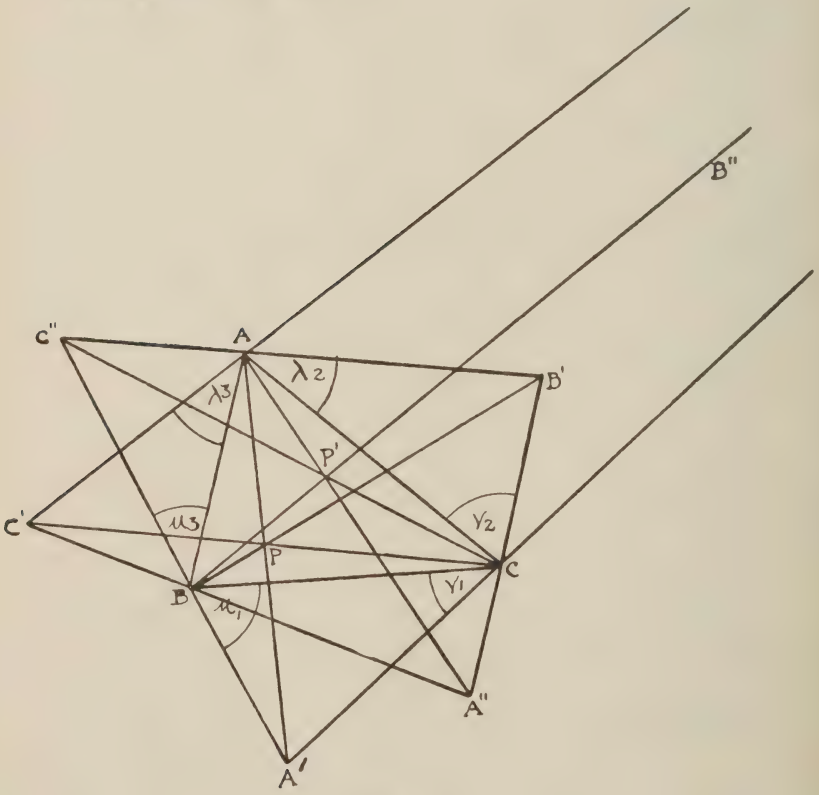

Fig. 1.

Using normal coordinates, then for AA′, $\frac{\beta}{\gamma} = \frac{\sin\mu_1/\sin(B+\mu_1)}{\sin\nu_1/\sin(C+\nu_1)}$,

∴ the lines AA′, BB′, CC′ are concurrent if

$$\frac{\sin\mu_1/\sin(B+\mu_1)}{\sin\nu_1/\sin(C+\nu_1)} \cdot \frac{\sin\nu_2/\sin(C+\nu_2)}{\sin\lambda_2/\sin(A+\lambda_2)} \cdot \frac{\sin\lambda_3/\sin(A+\lambda_3)}{\sin\mu_3/\sin(B+\mu_3)} = 1.$$

e.g. if $\mu_1 = \lambda_2 = 90 - C$, $\nu_1 = \lambda_3 = 90 - B$, $\nu_2 = \mu_3 = 90 - A$,

$$\frac{\beta}{\gamma} = \frac{\sin(90-C)/\sin(B+90-C)}{\sin(90-B)/\sin(C+90-B)} = \frac{\sec B}{\sec C},$$

and the lines are concurrent at the orthocentre $\sec A : \sec B : \sec C$. (The points A′, B′, C′ are the points in which the altitudes intersect the circumcircle of △ABC).

(2). If the angles at the vertices be interchanged, *e.g.* μ_1 and μ_3, then AA′, BB′, CC′ are concurrent if

$$\frac{\sin\mu_3/\sin(B+\mu_3)}{\sin\nu_2/\sin(C+\nu_2)} \cdot \frac{\sin\nu_1/\sin(C+\nu_1)}{\sin\lambda_3/\sin(A+\lambda_3)} \cdot \frac{\sin\lambda_2/\sin(A+\lambda_2)}{\sin\mu_1/\sin(B+\mu_1)} = 1,$$

which is the same condition as in (1). Hence if the lines are concurrent, they are concurrent when the angles are interchanged, *e.g.* interchanging the angles of example in (1), so that $A'BC = 90 - A$, $A'CB = 90 - A$, etc., then

$$\frac{\beta}{\gamma} = \frac{\sin(90-A)/\sin(B+90-A)}{\sin(90-A)/\sin(C+90-A)} = \frac{\cos(C-A)}{\cos(A-B)},$$

and the point of concurrence is $\cos(B-C) : \cos(C-A) : \cos(A-B)$, A′, B′, C′ being the images of the circumcentre in the sides.

(3). If C′BA′, etc., are straight lines, so that $\mu_1 + \mu_3 = 180 - B$, etc., and the lines be concurrent, then if the angles be interchanged as in (2), the two points of concurrence are isogonals.

e.g. if $\mu_1 = \nu_1 = A$, $\nu_2 = \lambda_2 = B$, $\lambda_3 = \mu_3 = C$, the point of concurrence is the Lemoine point $\sin A : \sin B : \sin C$, the points A′, B′, C′ being the points in which perpendiculars from the circumcentre O on the sides of the triangle meet the circles round OBC, OCA, OAB.

Interchanging the angles, so that $\mu_1 = \lambda_2 = C$, $\nu_1 = \lambda_3 = B$, $\nu_2 = \mu_3 = A$, the point of concurrence is the centroid

$$\operatorname{cosec} A : \operatorname{cosec} B : \operatorname{cosec} C,$$

the isogonal to the Lemoine point. The lines AA′, BB′, CC′ are bisected by the sides of the triangle.

(4). If $\mu_1 = \nu_1 = A \pm \theta$, $\nu_2 = \lambda_2 = B \pm \theta$, $\mu_3 = \lambda_3 = C \pm \theta$,

then $$\frac{\beta}{\gamma} = \frac{\sin(A \pm \theta)/\sin(B + A \pm \theta)}{\sin(A \pm \theta)/\sin(C + A \pm \theta)} = \frac{\sin(B \mp \theta)}{\sin(C \mp \theta)},$$

and the lines are concurrent at the point

$$\sin(A \mp \theta) : \sin(B \mp \theta) : \sin(C \mp \theta).$$

This point lies on the line joining the Lemoine point $\sin A : \sin B : \sin C$ to the circumcentre $\cos A : \cos B : \cos C$, for $\Sigma \sin(A \mp \theta)\sin(B - C) = 0$.

e.g. If $\theta = 0$, the point is the Lemoine point, and if $\theta = 60°$, the points are $\sin(A \mp 60) : \sin(B \mp 60) : \sin(C \mp 60)$, the two Isodynamic points.

(5). If all six angles $= \theta$, the point of concurrence is $\operatorname{cosec}(A + \theta) : \operatorname{cosec}(B + \theta) : \operatorname{cosec}(C + \theta)$, the isogonal to the point derived in (4) by making $\mu_1 = \nu_1 = A - \theta$, etc.

e.g. If $\theta = 60°$, the point is the Inner Isogonic point, the isogonal of one of the Isodynamic points.

(6). If $\mu_1 = \mu_3 = \mu$, $\nu_1 = \nu_2 = \nu$, $\lambda_2 = \lambda_3 = \lambda$, the lines AA′, BB′, CC′ are always concurrent, the point being

$$\frac{\sin\lambda}{\sin(A + \lambda)} : \frac{\sin\mu}{\sin(B + \mu)} : \frac{\sin\nu}{\sin(C + \nu)}.$$

e.g. If $\lambda = 90 - A$, $\mu = 90 - B$, $\nu = 90 - C$, the point of concurrence is the circumcentre $\cos A : \cos B : \cos C$, A′, B′, C′ lying on the circumcircle of $\triangle ABC$.

If $\lambda = 90 - C$, $\mu = 90 - A$, $\nu = 90 - B$, the point is

$$\frac{\cos C}{\cos(C - A)} : \frac{\cos A}{\cos(A - B)} : \frac{\cos B}{\cos(B - C)}.$$

(7). If P be a point within $\triangle ABC$, and if AP, BP, CP be produced to meet the circumcircles of $\triangle$s BPC, CPA, APB in A′, B′, C′, $\mu_1 = \mu_3$, $\nu_1 = \nu_2$, $\lambda_2 = \lambda_3$, and if these angles be λ, μ, ν, $\lambda + \mu + \nu = 180°$, and the $\angle$s BPC, CPA, APB in A′, B′, C′, $\mu_1 = \mu_3$, $\nu_1 = \nu_2$, $\lambda_2 = \lambda_3$, and if these angles be λ, μ, ν, $\lambda + \mu + \nu = 180°$, and the $\angle$s BPC, CPA, APB are $180 - \lambda$, $180 - \mu$, $180 - \nu$, for $A'BC = A'PC = C'PA = C'BA$, etc., and $BA'C = CPB = CAB = \lambda$, etc., and $BPC = 180 - \lambda$, etc.

e.g. If $\lambda = 90 - \frac{A}{2}$, $\mu = 90 - \frac{B}{2}$, $\nu = 90 - \frac{C}{2}$, the point P is the incentre 1 : 1 : 1, and the $\angle$s at P are $90 + \frac{A}{2}$, etc., A′, B′, C′ being the three excentres of $\triangle ABC$.

If $\lambda = A$, $\mu = B$, $\nu = C$, the point is the orthocentre.

(8). If the lines C′B, B′C be produced to meet in A″, A′C, C′A in B″, A′B, B′A in C″, and AA′ BB′, CC′ be concurrent, then AA″, BB″, CC″ are concurrent, for the condition for concurrence is

$$\frac{\sin(B+\mu_3)/\sin\mu_3}{\sin(C+\nu_2)/\sin\nu_2}\cdot\frac{\sin(C+\nu_1)/\sin\nu_1}{\sin(A+\lambda_3)/\sin\lambda_3}\cdot\frac{\sin(A+\lambda_2)/\sin\lambda_2}{\sin(B+\mu_2)/\sin\mu_1}=1,$$

the same condition as in (1); the point of concurrence is the isogonal to the point got by interchanging the angles μ_1 and μ_3, etc., as in (2).

e.g. If as in (1) $\mu_1 = \lambda_2 = 90 - C$, $\nu_1 = \lambda_3 = 90 - B$, $\nu_2 = \mu_3 = 90 - A$, the point of concurrence is $\sec(B - C) : \sec(C - A) : \sec(A - B)$, the isogonal to the point $\cos(B - C) : \cos(C - A) : \cos(A - B)$ derived in (2) by interchanging the angles.

(9). If A′B′C′ be regarded as the original triangle with triangles C′BA′, A′CB′, B′AC′ described on the sides, then if the lines AA′, BB′, CC′ are concurrent, B′B″, C′C″, A′A″ are concurrent, the points A″, B″, C″ being the points derived as in (8).

(10). If $\mu_1 = \mu_3 = \mu$, etc., the point of concurrence of AA″, BB″, CC″ is

$$\frac{\sin(A+\lambda)}{\sin\lambda} : \frac{\sin(B+\mu)}{\sin\mu} : \frac{\sin(C+\nu)}{\sin\nu},$$

and the lines AA″, BB″, CC″ are the isogonals of AA′, BB′, CC′.

e.g. If $\lambda = C$, $\mu = A$, $\nu = B$, AA′, BB′, CC′ meet in $\frac{\sin C}{\sin B} : \frac{\sin A}{\sin C} : \frac{\sin B}{\sin A}$ and AA″, BB″, CC″ in $\frac{\sin B}{\sin C} : \frac{\sin C}{\sin A} : \frac{\sin A}{\sin B}$, the two Brocard points.

If $\lambda = \mu = \nu = \theta$, AA′, BB′, CC′ meet in $\operatorname{cosec}(A + \theta) : \operatorname{cosec}(B + \theta) : \operatorname{cosec}(C + \theta)$ and AA″, BB″, CC″ in $\sin(A + \theta) : \sin(B + \theta) : \sin(C + \theta)$, the same point as was derived in (4) by making $\mu_1 = \nu_1 = A - \theta$, etc.

(11). If $\lambda + \mu + \nu = 180°$, and AA″, BB″, CC″ meet in P′, BP′CA″, etc., are concyclic, for $BP'C = A + \lambda$, and $BA''C = 180 - A - \lambda$.

Again, $BAB' = A + \lambda$ and $BA''B' = BA''C = 180 - A - \lambda$, ∴ BAB′A″ and the other five sets of corresponding points are concyclic.

e.g. If $\lambda = \mu = \nu = 60°$, P′ is the isogonal of the Inner Isogonic point and $BP'C = A + 60$.

(12). If the triangles BA′C, CB′A, AC′B be described internally on the sides, AA′, BB′, CC′ are concurrent if

$$\frac{\sin\mu_1/\sin(B-\mu_1)}{\sin\nu_1/\sin(C-\nu_1)}\cdot\frac{\sin\nu_2/\sin(C-\nu_2)}{\sin\lambda_2/\sin(A-\lambda_2)}\cdot\frac{\sin\lambda_3/\sin(A-\lambda_3)}{\sin\mu_3/\sin(B-\mu_3)}=1.$$

As in the case of the external triangles, the lines are concurrent if μ_1 and μ_3, etc., be interchanged, and if $\mu_1=\mu_3=\mu$, $\nu_1=\nu_2=\nu$, $\lambda_2=\lambda_3=\lambda$, the lines are always concurrent, the point being

$$\frac{\sin\lambda}{\sin(A-\lambda)}:\frac{\sin\mu}{\sin(B-\mu)}:\frac{\sin\nu}{\sin(C-\nu)}.$$

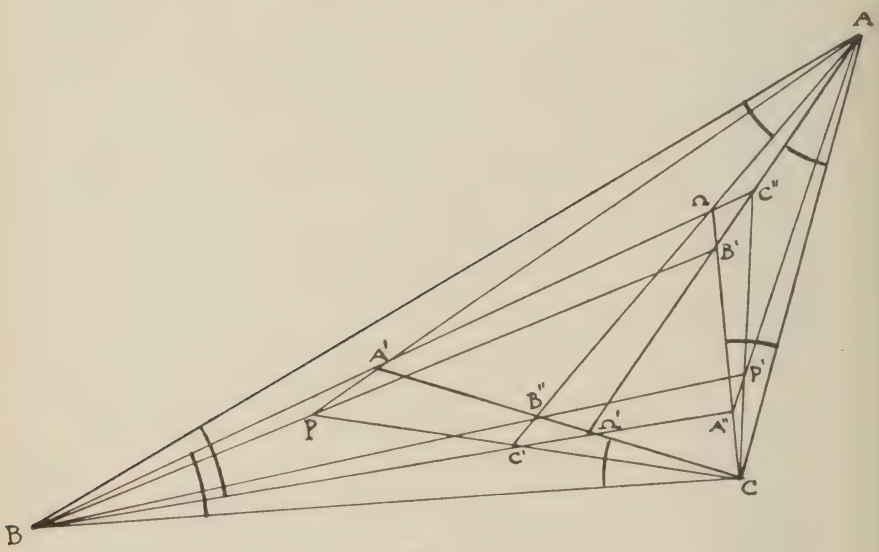

Fig. 2.

e.g. If $\mu_1=\mu_3=90-A$, $\nu_1=\nu_2=90-B$, $\lambda_2=\lambda_3=90-C$, the point of concurrence is $\frac{\cos C}{\cos B}:\frac{\cos A}{\cos C}:\frac{\cos B}{\cos A}$.

If $\lambda+\mu+\nu=180$, then as in the case of the external triangles, A′BPC, etc., are concyclic points, the angles between the lines AP, BP, CP being λ, μ, ν.

e.g. If $\lambda=\mu=\nu=60°$, the point is

$$\operatorname{cosec}(A-60):\operatorname{cosec}(B-60):\operatorname{cosec}(C-60),$$

the Outer Isogonic point.

(13). If CB′, BC′ be produced to meet in A″, etc., then if AA′, BB′, CC′ be concurrent, so are AA″, BB″, CC″, and the point of concurrence is the isogonal of the point got by interchanging the angles μ_1 and μ_3, etc., just as in the case of the external mentioned in (8). A′A″, B′B″, C′C″ are also concurrent as in (9).

e.g. If $\mu_1 = \mu_3 = \frac{B}{3}$, $\nu_1 = \nu_2 = \frac{C}{3}$, $\lambda_2 = \lambda_3 = \frac{A}{3}$, AA′, BB′, CC′ meet in the point $\sec\frac{A}{3} : \sec\frac{B}{3} : \sec\frac{C}{3}$ and AA″, BB″, CC″ in the point $\cos\frac{A}{3} : \cos\frac{B}{3} : \cos\frac{C}{3}$. The first point with reference to triangle A′B′C′ can be shown to be

$$\frac{\sin(60 - A/3)}{\sin(60 + A/3)} : \frac{\sin(60 - B/3)}{\sin(60 + B/3)} : \frac{\sin(60 - C/3)}{\sin(60 + C/3)}.$$

e.g. If all six angles equal the Brocard angle ω (Fig. 2), AA′, BB′, CC′ meet in the point $\operatorname{cosec}(A - \omega) : \operatorname{cosec}(B - \omega) : \operatorname{cosec}(C - \omega)$, and AA″, BB″, CC″ in the isogonal point $\sin(A - \omega) : \sin(B - \omega) : \sin C - \omega)$, a point lying on the line joining the Lemoine point to the circumcentre. The points A′, B′, C′ and A″, B″, C″ form the Brocard triangles.

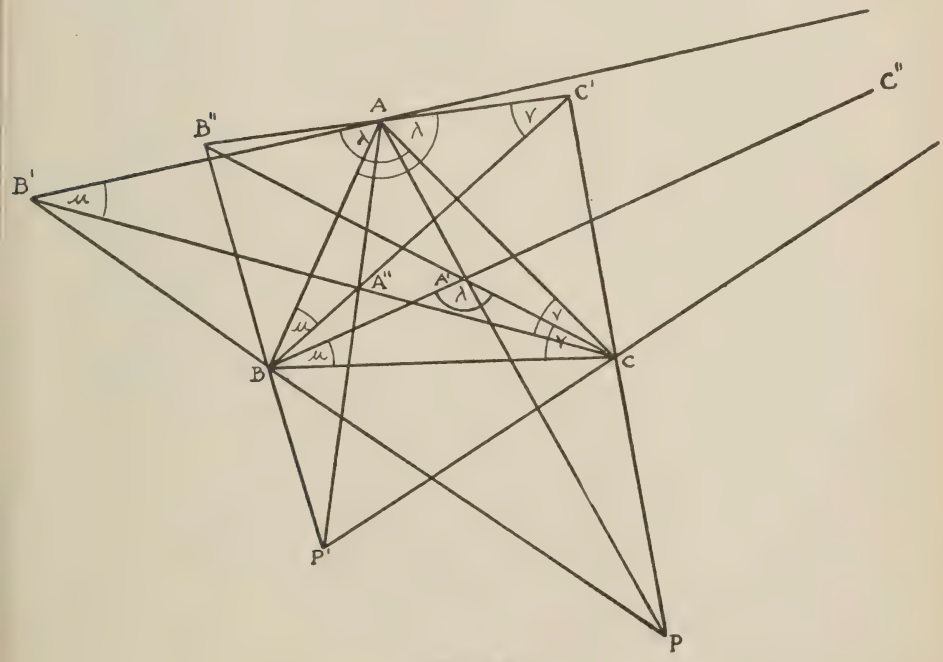

Fig. 13.

(14). If (Fig. 3) $\mu_1 = \mu_3 = \mu$, $\nu_1 = \nu_2 = \nu$, $\lambda_2 = \lambda_3 = \lambda$, and if $\lambda + \mu + \nu = 180°$, and AA′, BB′, CC′ meet in P, and AA″, BB″, CC″ in P′, then as in case of the external triangles mentioned in (10), the following sets of four points are concyclic, BP′CA″, CP′AB″, AP′BC″, CAC′A″, CBC′B″, ABA′B″, ACA′C″, BCB′C″, BAB′A″.

e.g. If $\lambda = \mu = \nu = 60$, P′ is $\sin(A - 60) : \sin(B - 60) : \sin(C - 60)$, one of the Isodynamic points ; see (4).

On the continued fractions which represent the functions of Hermite and other functions defined by differential equations.

By Professor E. T. Whittaker, F.R.S.

(*Read and Received 9th January 1914*).

§ 1. *Introduction.*

The functions of Hermite, which are the same as the functions associated with the parabolic cylinder in harmonic analysis, may be defined* by the differential equation which they satisfy, namely,

$$\frac{d^2y}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where n denotes any constant.

The standard solution of the equation, which is denoted by $\mathrm{D}_n(z)$, may be represented by the asymptotic expansion

$$\mathrm{D}_n(z) = e^{-\frac{1}{4}z^2} z^n \left\{ 1 - \frac{n(n-1)}{2z^2} + \frac{n(n-1)(n-2)(n-3)}{2 \,.\, 4 \,.\, z^4} - \ldots \right\}; \quad \ldots\ldots (2)$$

and a second independent solution of the differential equation is $\mathrm{D}_{-n-1}(iz)$, so that the complete solution is of the form

$$y = a\mathrm{D}_n(z) + b\mathrm{D}_{-n-1}(iz)$$

where a and b are arbitrary constants.

In § 2 of the present paper it is shown that each of these solutions $\mathrm{D}_n(z)$ and $\mathrm{D}_{-n-1}(iz)$ can be calculated by means of a continued fraction. In § 3 it is shown that the two continued

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* *Cf.* Hermite, *Comptes Rendus 58* (1864), pp. 93, 266.
Whittaker, *Proc. Lond. Math. Soc. 35* (1903), p. 417.
Myller-Lebedeff, *Math. Ann. 64* (1907), p. 388.
Watson, *Proc. Lond. Math. Soc.* (2) *8* (1910), p. 393.
Curzon, *Proc. Lond. Math. Soc.* (2) *12* (1912), p. 236.

fractions thus introduced bear to each other a relation similar to that between the two C.F.'s which represent the roots of a quadratic equation. In § 4 it is shown that there exist four other continued fractions connected with the Hermite functions, and that these are suited for calculation when z is small, the former C.F.'s being suited for calculation when z is large. In § 5 it is shown that other properties of the Hermite functions can be derived readily from the continued fractions. In § 6 it is explained why the C.F.'s represent certain particular solutions of the differential equation rather than others: and finally, in § 7 the results are extended to a much more general class of functions, which are defined by linear differential equations of the second order, and include many of the functions required in the applications of analysis.

§ 2. *Two continued fractions associated with the Hermite functions.*

In the differential equation (1) write

$$y = e^{-\frac{1}{4}z^2} u ;$$

it becomes

$$u'' - zu' + nu = 0.$$

Differentiating this, we obtain

$$u''' - zu'' + (n-1)u' = 0.$$

Differentiating again, we obtain

$$u'''' - zu''' + (n-2)u'' = 0,$$

and so on; this series of recurrence-equations at once gives formally the continued fraction

$$\frac{u'}{u} = \frac{n}{z-}\ \frac{n-1}{z-}\ \frac{n-2}{z-}\ \frac{n-3}{z-} \quad \text{.....(A)}$$

If, on the other hand, we integrate the differential equation instead of differentiating it, we have (denoting $\int u dx$ by u_1, $\int u_1 dx$ by u_2, etc.)

$$u' - zu + (n+1)u_1 = 0,$$
$$u - zu_1 + (n+2)u_2 = 0,$$

and so on ; this series of recurrence-equations gives formally the continued fraction

$$\frac{u'}{u} = z - \frac{n+1}{z-} \quad \frac{n+2}{z-} \quad \frac{n+3}{z-} \quad \ldots\ldots\ldots\ldots\ldots\ldots (B)$$

This method of solving differential equations is due to Euler. Its subsequent neglect is not easily explicable, for its simplicity and directness are very attractive ; the methods of solution by power-series and by definite integrals appear forced and artificial in comparison.

The question now is, which of the recognised solutions of the differential equation in power-series correspond to the continued fractions (A) and (B). This question is settled by applying the usual method of converting continued fractions into series ; we thus obtain from (A)

$$\frac{u'}{u} = \frac{n}{z} + \frac{n(n-1)}{z^3} + \frac{n(n-1)(2n-3)}{z^5} + \ldots\ldots\ldots\ldots (A')$$

while from (B) we obtain

$$\frac{u'}{u} = z - \frac{n+1}{z} - \frac{(n+1)(n+2)}{z^3} - \ldots\ldots\ldots\ldots\ldots (B')$$

Integrating, we obtain from (A')

$$u = z^n \left\{ 1 - \frac{n(n-1)}{2z^2} + \ldots \right\} \ldots\ldots\ldots\ldots\ldots\ldots (A'')$$

while from (B') we obtain

$$u = e^{\frac{1}{2}z^2} z^{-n-1} \left\{ 1 + \frac{(n+1)(n+2)}{2z^2} + \ldots \right\} \ldots\ldots\ldots\ldots (B'')$$

Comparing (A'') and (B'') with equation (2), we see that (A'') may be written

$$u = e^{\frac{1}{4}z^2} D_n(z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A''')$$

while (B'') may be written

$$u = i^{n+1} e^{\frac{1}{4}z^2} D_{-n-1}(iz). \ldots\ldots\ldots\ldots\ldots\ldots (B''')$$

We see therefore that *the two continued fractions* (A) *and* (B), *which are associated with the differential equation* (1) *correspond respectively to the two independent solutions* $D_n(z)$ *and* $D_{-n-1}(iz)$.

These continued fractions are well suited to numerical computation when z is large, and can be used for this purpose directly, without the reduction to series-form and the subsequent integration. They have, moreover, the advantage over series that the approximation is more rapid—a well-known property of continued fractions, depending ultimately on the fact that in a C.F. both the numerators and the denominators of the convergents are helping on the approximation, whereas a series is (so to speak) all numerator, and so is deprived of the assistance which might be afforded by a denominator; for this reason a C.F. convergent which includes only the n^{th} power of a variable in its numerator and denominator will generally give as good an approximation as a series which is continued as far as the terms involving the $2n^{\text{th}}$ power of the variable. In our case the continued fraction (A) truncated at its third term

$$\frac{n}{z-}\ \frac{n-1}{z-}\ \frac{n-2}{z}$$

is equivalent to a series carried as far as the term in $1/z^6$.

§ 3. *Analogy with the solution of a quadratic equation.*

The above result presents an analogy with the solution of an ordinary quadratic equation by continued fractions; if, for example, we consider the equation

$$x^2 - 3x + 2 = 0,$$

we can derive from it the two continued fractions

$$x = 3 - \tfrac{2}{3} - \tfrac{2}{3} - \tfrac{2}{3} - \cdots$$

and

$$x = \tfrac{2}{3} - \tfrac{2}{3} - \tfrac{2}{3} - \cdots$$

of which the first has the value 2 and the second has the value 1; and these are the two roots of the quadratic. The resemblance with the work above becomes evident if we reflect that these are precisely the two C.F.'s which we should obtain if we were to solve the differential equation

$$\frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = 0$$

by the method of § 2.

§ 4. *Four other continued fractions associated with the Hermite functions.*

The expansion (2) for the function $D_n(z)$ is an asymptotic expansion proceeding in *descending* powers of z, and therefore

useful for calculation when z is large. If, on the other hand, z is not large, we require series which proceed in *ascending* powers of z. It is easy to show that one solution of the differential equation (1) is the series

$$E_n(z) = e^{-\frac{1}{4}z^2}\left\{1 - \frac{n}{2!}z^2 + \frac{n(n-2)}{4!}z^4 - \ldots\right\} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

or

$$E_n(z) = e^{\frac{1}{4}z^2}\left\{1 - \frac{n+1}{2!}z^2 + \frac{(n+1)(n+3)}{4!}z^4 - \ldots\right\} \quad \ldots\ldots(4)$$

and another solution is

$$O_n(z) = e^{-\frac{1}{4}z^2}\left\{z - \frac{n-1}{3!}z^3 + \frac{n-1\,.\,n-3}{5!}z^5 - \ldots\right\} \quad \ldots\ldots\ldots(5)$$

$$= e^{\frac{1}{4}z^2}\left\{z - \frac{n+2}{3!}z^3 + \frac{n+2\,.\,n+4}{5!}z^5 - \ldots\right\} \quad \ldots\ldots\ldots\ldots(6)$$

The standard solution $D_n(z)$ may be expressed in terms of these by the equation

$$D_n(z) = \frac{2^{\frac{n}{2}}\pi^{\frac{1}{2}}}{\Gamma\left(-\frac{n-1}{2}\right)}E_n(z) - \frac{2^{\frac{n+1}{2}}\pi^{\frac{1}{2}}}{\Gamma\left(-\frac{n}{2}\right)}O_n(z).$$

We shall now find continued fractions which correspond to the solutions $E_n(z)$ and $O_n(z)$.

If in (1) we write

$$z^2 = x, \quad y = ue^{-\frac{1}{4}x},$$

the equation becomes

$$4x\frac{d^2u}{dx^2} + (-2x+2)\frac{du}{dx} + nu = 0,$$

and obtaining a continued fraction by successive differentiation of this equation, we have

$$\frac{u'}{u} = \frac{n}{2x-2-}\ \frac{4(n-2)x}{2x-6-}\ \frac{4(n-4)x}{2x-10-}\ ; \quad \ldots\ldots\ldots\ldots\ldots(C)$$

if, on the other hand, we obtain a continued fraction by successive integration of the same equation, it is found to be

$$2x\frac{u'}{u} = 1 + x - \frac{(n+2)x}{3+x-}\ \frac{(n+4)x}{5+x-} \quad \ldots\ldots\ldots\ldots\ldots(D)$$

Again, if in (1) we write $z^2 = x$, $y = we^{\frac{1}{2}x}$, the equation becomes

$$4x\frac{d^2w}{dx^2} + (2x+2)\frac{dw}{dx} + (n+1)w = 0.$$

By successive differentiation of this equation we obtain the continued fraction

$$\frac{w'}{w} = -\frac{n+1}{2x+2-}\ \frac{4(n+3)x}{2x+6-}\ \frac{4(n+5)x}{2x+10-} \quad \ldots\ldots\ldots\ldots\ldots(E)$$

while by successive integration of the equation we obtain

$$2x\frac{w'}{w} = 1 - x - \frac{(n-1)x}{3 - x -} \quad \frac{(n-3)x}{5 - x -} \quad \ldots\ldots\ldots\ldots\ldots\ldots (F)$$

It may easily be shown by the methods of § 2 that *the continued fractions* (C), (D), (E), (F) *correspond respectively to the series* (3), (4), (5), (6); thus whether z be large or small, the computation of the parabolic-cylinder functions can be performed by means of continued fractions.

Taking as an example $n = \frac{1}{2}$, $z = \frac{1}{2}$, the continued fraction (C) furnishes the value of u'/u correctly to the first, second, third, or fourth place of decimals according as the first, second, third, or fourth convergent is used. Mr A. Milne has kindly calculated for me the following table:

z	$\frac{1}{u}\frac{du}{dx}$	$\frac{1}{u}\frac{du}{dz}$
0	− 0·2500	− 0·0000
0·1	− 0·2513	− 0·0503
0·2	− 0·2551	− 0·1020
0·3	− 0·2617	− 0·1570
0·4	− 0·2714	− 0·2171
0·5	− 0·2847	− 0·2847
0·6	− 0·3025	− 0·3630
0·7	− 0·3260	− 0·4564
0·8	− 0·3569	− 0·5710
0·9	− 0·3983	− 0·7169
1·0	− 0·4548	− 0·9097

The second column gives the value of $\frac{1}{u}\frac{du}{dx}$, computed from the continued fraction (C) ; and the third column gives the corresponding value of $\frac{1}{u}\frac{du}{dz}$, which is obtained by multiplying the number in the second column by $2z$. From this we can derive $\Delta \log u$, then $\log u$, and finally u itself, if this be the form in which the function is required.

§5. *Deduction of the recurrence-formulae for the Hermite functions.*

It is easy to deduce from the continued fractions the characteristic properties of the Hermite functions. For example, we see at once that the continued fraction

$$\frac{n-1}{z-}\quad\frac{n-2}{z-}\quad\frac{n-3}{z-}\cdots$$

represents both $\frac{u'_{n-1}}{u_{n-1}}$ and $\frac{u_n''}{u_n'}$, where u_{n-1} denotes the same function of the parameter $(n-1)$ that u_n is of n ; and therefore we must have

$$\frac{u_n''}{u_n'} = \frac{u'_{n-1}}{u_{n-1}}$$

or

$$u_n' = \mathrm{C}u_{n-1}$$

where C is independent of z. Making z very large in (A''), we see that the constant C has the value n ; so we have

$$u_n' = nu_{n-1},$$

which can be written

$$\frac{d\mathrm{D}_n(z)}{dz} + \tfrac{1}{2}z\,\mathrm{D}_n(z) - n\,\mathrm{D}_{n-1}(z) = 0.$$

This is one of the recurrence-formulae satisfied by the Hermite function $\mathrm{D}_n(z)$.

§6. *The selection of particular solutions from the general solution, in the continued-fraction method.*

We are naturally led to enquire how the method of solving a differential equation by a continued fraction leads in every case to

some particular solution of the equation instead of to the general solution; for it does not at first sight seem evident where the restriction to particularity is introduced in the process of solution.

To discuss this question, let us consider a simple equation such as

$$\frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = 0,$$

and suppose the process of solution arrested at the r^{th} step, so that we obtain the finite continued fraction

$$\tfrac{2}{3} - \tfrac{2}{3} - \tfrac{2}{3} - \cdots - \tfrac{2}{3} - \frac{y^{(r+1)}}{y^{(r)}}.$$

This continued fraction represents y'/y if y is *any solution whatever* of the equation; so that if for $y^{(r+1)}/y^{(r)}$ in the final quotient we substitute the value 1 (corresponding to the solution $y = e^x$), we get the value 1 for the entire continued fraction; and if for $y^{(r+1)}/y^{(r)}$ we substitute the value 2 (corresponding to the solution $y = e^{2x}$), we get the value 2 for the entire continued fraction. The question now arises, what value has the entire C.F. when we assign to $y^{(r+1)}/y^{(r)}$ in the final quotient a value which is neither 1 nor 2. To determine this, first take $y^{(r+1)}/y^{(r)} = 1 + \epsilon$, where ϵ is supposed small, so that its square can be neglected, then the entire continued fraction has the value $1 + 2^{-r}\epsilon$; so that if to the last quotient we assign a value near to unity, the entire C.F. will have a value nearer still to unity. On the other hand, if we take $y^{(r+1)}/y^{(r)} = 2 + \epsilon$, where ϵ is small, then the entire C.F. has the value $2 + 2^r\epsilon$; so that if to the last quotient we assign a value near to 2, the entire C.F. will have a value much more remote from 2.

The two values 1 and 2 of the entire C.F. may therefore, by analogy with the theory of orbits in dynamics, be called the *stable* and *unstable* values respectively; and it is in consequence of its instability that the value 2 drops out.

§ 7. *Extension of the preceding results to more general functions.*

The functions of Hermite are included* in an extensive class

* *Cf.* a paper by the present writer in *Bull. Amer. Math. Soc.* (2) *10* (1903), p. 125.

of functions denoted by the symbol $W_{k,m}(z)$; this class of functions comprehends also the error-function, the incomplete-gamma-function, the logarithm-integral, the cosine-integral, and the Bessel functions.

It will now be shown that *the results of* § 4 *can be extended to these functions* $W_{k,m}(z)$.

For these functions satisfy the differential equation

$$\frac{d^2u}{dz^2}+\left(-\tfrac{1}{4}+\frac{k}{z}+\frac{\frac{1}{4}-m^2}{z^2}\right)u=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

This equation has the solutions

$$M(z)=z^{\frac{1}{2}+m}e^{\frac{1}{2}z}\left\{1-\frac{\frac{1}{2}+m+k}{1\,.\,2m+1}z+\frac{\frac{1}{2}+m+k\,.\,\frac{3}{2}+m+k}{1\,.\,2\,.\,2m+1\,.\,2m+2}z^2-\ldots\right\} \ldots\ldots(8)$$

$$=z^{\frac{1}{2}+m}e^{-\frac{1}{2}z}\left\{1+\frac{\frac{1}{2}+m-k}{1\,.\,2m+1}z+\frac{\frac{1}{2}+m-k\,.\,\frac{3}{2}+m-k}{1\,.\,2\,.\,2m+1\,.\,2m+2}z^2+\ldots\right\} \ldots(9)$$

and

$$P(z)=z^{\frac{1}{2}-m}e^{-\frac{1}{2}z}\left\{1+\frac{\frac{1}{2}-m-k}{1\,.\,-2m+1}z+\frac{\frac{1}{2}-m-k\,.\,\frac{3}{2}-m-k}{1\,.\,2\,.\,-2m+1\,.\,-2m+2}z^2+\ldots\right\}$$
$$\ldots\ldots\ldots\ldots(10)$$

$$=z^{\frac{1}{2}-m}e^{\frac{1}{2}z}\left\{1-\frac{\frac{1}{2}-m+k}{1\,.\,-2m+1}z+\frac{\frac{1}{2}-m+k\,.\,\frac{3}{2}-m+k}{1\,.\,2\,.\,-2m+1\,.\,-2m+2}z^2-\ldots,\right\}$$
$$\ldots\ldots\ldots\ldots(11)$$

and the standard solution $W_{k,m}(z)$ is expressible in terms of these two solutions by the formula

$$W_{k,m}(z)=\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}M(z)+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}P(z).$$

Now writing $u=yz^{m+\frac{1}{2}}e^{\frac{1}{2}z}$, the differential equation (7) becomes

$$y\frac{d^2y}{dz^2}+(2m+1+z)\frac{dy}{dz}+(k+m+\tfrac{1}{2})y=0.$$

By successive differentiation of this, as in § 2, we obtain the continued fraction

$$\frac{y'}{y}=-\frac{k+m+\frac{1}{2}}{2m+1+z}-\frac{(k+m+\frac{3}{2})z}{2m+2+z}-\frac{(k+m+\frac{5}{2})z}{2m+3+z}-\ldots$$

Moreover, since the differential equation (7) is unaltered when m is changed to $-m$, another continued fraction (corresponding to a second solution of the differential equation) can be obtained by changing the sign of m in this ; and since the equation (7) is also unaltered when the signs of k and z are reversed simultaneously, we can obtain two more continued fractions from these by reversing the signs of k and z. These C.F.'s correspond respectively to the series (8), (9), (10), (11) above.

Thus we obtain altogether 4 continued fractions associated with the function $W_{k,m}(z)$; these correspond exactly to the 4 C.F.'s obtained in § 4 for the Hermite functions. By taking special values for k and m, we can deduce various continued fractions for the error-function, cosine-integral, etc.

On the general solution of Mathieu's equation.

By Professor E. T. Whittaker, F.R.S.

(*Read 12th June 1914. Received 5th July 1914*).

§ 1. *Description of paper.*

The differential equation of Mathieu, or "equation of the elliptic cylinder functions,"

$$\frac{d^2y}{dz^2} + (a + 16q\cos 2z)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

occurs in many physical and astronomical problems. From the general theory of linear differential equations, we learn that its solution is of the type

$$y = Ae^{\mu z}\phi(z) + Be^{-\mu z}\psi(z)$$

where A and B denote arbitrary constants, μ is a constant depending on the constants a and q of the differential equation, and $\phi(z)$ and $\psi(z)$ are periodic functions of z. For certain values of a and q the constant μ vanishes, and the solution y is then a purely periodic function of z; but in general μ is different from zero.

While the general character of the solution from the function-theory point of view is thus known, its actual analytical determination presents great difficulties. The chief impediment is that the constant μ cannot readily be found in terms of a and q. It was in order to determine μ that Hill invented the celebrated method known by his name, which introduced infinite determinants into analysis. But it is much to be desired that μ should be found in some more manageable form than as a root of a determinant with an infinite number of rows and columns.

It has occurred to me as possible that the difficulty may be overcome by introducing another parameter in place of a. This suggestion may be illustrated by reference to a well-known feature of the theory of elliptic functions. In elliptic functions the parameter used by the earlier workers was the modulus k; but

expansions in terms of k are awkward and unmanageable, and it was shown by Jacobi that the true parameter of the subject is the quantity $exp\left(-\frac{\pi K'}{K}\right)$: the introduction of this made possible the theory of the theta-functions and q-series, which furnish the most satisfactory expansions of elliptic functions.

In the present paper it is shown that the solution of Mathieu's equation (1) can be effected by introducing in place of a a new parameter which will be denoted by σ. The parameter μ whose value is required, and the parameter a itself, will be expressed in terms of σ and the parameter q: so that when a and q are given we can first find σ from them, and then find μ from σ and q, and ultimately obtain the solution y of the equation.

§ 2. *The solution.*

As already mentioned, there are certain values of a for which the solution of Mathieu's equation is purely periodic; in particular, if

$$a = 1 + 8q - 8q^2 - 8q^3 + \dots$$

then the equation (1) has a solution

$$y = \sin z + q\sin 3z + q^2(\sin 3z + \tfrac{1}{3}\sin 5z) + q^3(\tfrac{1}{3}\sin 3z + \tfrac{4}{9}\sin 5z + \tfrac{1}{1\,8}\sin 7z) + \dots;$$

and if

$$a = 1 - 8q - 8q^2 + 8q^3 + \dots$$

then the equation (1) has a solution

$$y = \cos z + q\cos 3z + q^2(-\cos 3z + \tfrac{1}{3}\cos 5z) + q^3(\tfrac{1}{3}\cos 3z - \tfrac{4}{9}\cos 5z + \tfrac{1}{1\,8}\cos 7z) + \dots$$

The form of these series suggests that they may be degenerate cases of a general solution of Mathieu's equation, having the form

$$y = e^{\mu z}u$$

where

$$u = \sin(z-\sigma) + a_3\cos(3z-\sigma) + b_3\sin(3z-\sigma) + a_5\cos(5z-\sigma) + b_5\sin(5z-\sigma) + \dots,$$

where σ is a new parameter. The first of the two special solutions above would then correspond to $\sigma = 0$, and the second of them to $\sigma = \frac{\pi}{2}$.

It will be observed that there is no term in $\cos(z-\sigma)$; this really constitutes the definition of σ, and the possibility of obtaining series which remain convergent for all real values of σ

depends on our choosing σ in this way. The coefficient of $\sin(z-\sigma)$ is taken to be unity, which amounts to fixing the arbitrary constant by which the solution is multiplied.

Let us then try to satisfy the differential equation (1) by an expression of this form, the parameters μ and a being expressed in terms of σ and q by series proceeding in ascending powers of q, thus :—

$$\mu = q\kappa(\sigma) + q^2\lambda(\sigma) + q^3\mu(\sigma) + q^4\nu(\sigma) + \dots$$
$$a = 1 + q\alpha(\sigma) + q^2\beta(\sigma) + q^3\gamma(\sigma) + q^4\delta(\sigma) + \dots$$

We shall write

$$u = \sin(z-\sigma) + q\mathrm{A}(z,\sigma) + q^2\mathrm{B}(z,\sigma) + q^3\mathrm{C}(z,\sigma) + \dots$$

where, as we have seen, there are to be no terms in $\sin(z-\sigma)$ or $\cos(z-\sigma)$ in $\mathrm{A}(z,\sigma)$, $\mathrm{B}(z,\sigma)$, $\mathrm{C}(z,\sigma)$, ...

Substituting these expansions in the differential equation, we obtain

$$-\sin(z-\sigma) + q\mathrm{A}'' + q^2\mathrm{B}'' + q^3\mathrm{C}'' + \dots$$
$$+ 2(q\kappa + q^2\lambda + q^3\mu + \dots)\{\cos(z-\sigma) + q\mathrm{A}' + q^2\mathrm{B}' + q^3\mathrm{C}' + \dots\}$$
$$+ \{(q\kappa + q^2\lambda + q^3\mu + \dots)^2 + 1 + q\alpha + q^2\beta + q^3\gamma + \dots + 16q\cos 2z\}$$
$$\{\sin(z-\sigma) + q\mathrm{A} + q^2\mathrm{B} + q^3\mathrm{C} + \dots\} = 0.$$

Equating to zero the terms independent of q in this equation, we have a mere identity. Equating to zero the terms involving the first power of q, we have

$$\mathrm{A}'' + \mathrm{A} + 2\kappa\cos(z-\sigma) + \alpha\sin(z-\sigma) + 16\cos 2z\sin(z-\sigma) = 0$$

or

$$\mathrm{A}'' + \mathrm{A} + (2\kappa - 8\sin 2\sigma)\cos(z-\sigma) + (\alpha - 8\cos 2\sigma)\sin(z-\sigma) + 8\sin(3z-\sigma) = 0.$$

Since A is not to contain any terms in $\cos(z-\sigma)$ or $\sin(z-\sigma)$, this gives at once the three equations

$$\alpha = 8\cos 2\sigma$$
$$\kappa = 4\sin 2\sigma$$
$$\mathrm{A} = \sin(3z-\sigma).$$

Next equating to zero the terms involving the square of q, we have

$$\mathrm{B}'' + \mathrm{B} + 2\kappa\mathrm{A}' + 2\lambda\cos(z-\sigma) + \kappa^2\sin(z-\sigma) + \alpha\mathrm{A} + \beta\sin(z-\sigma) + 16\mathrm{A}\cos 2z = 0$$

or

$$\mathrm{B}'' + \mathrm{B} + 2\lambda\cos(z-\sigma) + (16\sin^2 2\sigma + 8 + \beta)\sin(z-\sigma) + 24\sin 2\sigma\cos(3z-\sigma) + 8\cos 2\sigma\sin(3z-\sigma) + 8\sin(5z-\sigma) = 0.$$

Applying the conditions that B is not to contain any terms in $\sin(z-\sigma)$ or $\cos(z-\sigma)$, this gives

$$\lambda = 0$$

$$\beta = -16 + 8\cos 4\sigma$$

$$B = 3\sin 2\sigma\cos(3z-\sigma) + \cos 2\sigma\sin(3z-\sigma) + \tfrac{1}{3}\sin(5z-\sigma).$$

The coefficients of the higher powers of q in the series can be found in the same way.

We thus obtain the following theorem :

Mathieu's equation (1) *is satisfied by a function which we shall denote by* $\Lambda(z)$, *which can be expressed in the form*

$$\Lambda(z) = e^{\mu z}u(z),$$

where $u(z)$ *is a purely periodic function of* z, *and* μ *is given by the power-series in* q

$$\mu = 4q\sin 2\sigma - 12q^3\sin 2\sigma - 12q^4\sin 4\sigma + \dots ;$$

in this equation σ *is a parameter connected with the parameters* a *and* q *of the differential equation by the relation*

$$a = 1 + 8q\cos 2\sigma + (-16 + 8\cos 4\sigma)q^2 - 8q^3\cos 2\sigma + (\tfrac{256}{3} - 88\cos 4\sigma)q^4 + \dots ;$$

and $u(z)$ *is given by the Fourier series*

$$u(z) = \sin(z-\sigma) + a_3\cos(3z-\sigma) + b_3\sin(3z-\sigma) + a_5\cos(5z-\sigma) + b_5\sin(5z-\sigma) + a_7\cos(7z-\sigma) + \dots$$

where the coefficients are given by the power-series in q

$$b_3 = q + q^2\cos 2\sigma + (-\tfrac{14}{3} + 5\cos 4\sigma)q^3 + (-\tfrac{74}{9}\cos 2\sigma + 7\cos 6\sigma)q^4 + \dots$$

$$a_3 = 3q^2\sin 2\sigma + 3q^3\sin 4\sigma + (-\tfrac{274}{9}\sin 2\sigma + 9\sin 6\sigma)q^4 + \dots$$

$$b_5 = \tfrac{1}{3}q^2 + \tfrac{4}{9}q^3\cos 2\sigma + (-\tfrac{1556}{54} + \tfrac{82}{7}\cos 4\sigma)q^4 + \dots$$

$$a_5 = \tfrac{140}{9}q^3\sin 2\sigma + \tfrac{446}{7}q^4\sin 4\sigma + \dots$$

$$b_7 = \tfrac{1}{18}q^3 + \tfrac{1}{12}q^4\cos 2\sigma + \dots$$

$$a_7 = \tfrac{3508}{1008}q^4\sin 2\sigma + \dots$$

$$b_9 = \tfrac{1}{1800}q^4 + \dots$$

.....

Moreover, since a is an even function of σ, it is evident that *a second solution of the equation is obtained by merely changing* σ *to* $-\sigma$ *in the above formulae*: so that *the general solution of Mathieu's equation* (1) *is*

$$y = A\Lambda(z, \sigma, q) + B\Lambda(z, -\sigma, q)$$

where A *and* B *are arbitrary constants.*

When σ has either of the special values 0 and $\frac{\pi}{2}$, μ vanishes and one solution of Mathieu's equation is purely periodic: the solution $\Lambda(z, -\sigma, q)$ then ceases to be distinct from $\Lambda(z, \sigma, q)$, and we have the "logarithmic case" as with Bessel functions.

The above series are better adapted for numerical calculation than the infinite determinant.

§ 3. *The general terms of the series.*

Although for purposes of practical computing it is not a matter of much importance to have expressions for the general terms of the series, it may be of interest to determine some of them.

We shall first obtain the constants μ and a in a finite form in terms of a_3 and b_3. Substitute

$$y = e^{\mu z}\{\sin(z-\sigma) + a_3\cos(3z-\sigma) + b_3\sin(3z-\sigma) + \ldots\}$$

in the differential equation (1), and equate to zero the coefficient of $\cos(z-\sigma)$ in the equation so obtained: we thus find

$$\mu = 4q\sin 2\sigma - 4qa_3.$$

Similarly equating to zero the coefficient of $\sin(z-\sigma)$, we have

$$a = 1 + 8q\cos 2\sigma - \mu^2 - 8qb_3.$$

Thus μ and a are expressed in a finite form in terms of a_3 and b_3.

Next we shall show how the coefficients a_{2r+1} and b_{2r+1} may themselves be determined. It is obvious from what has preceded that b_{2r+1} and a_{2r+1} will be of the form

$$b_{2r+1} = A_r q^r + B_r q^{r+1}\cos 2\sigma + (C_r + D_r\cos 4\sigma)q^{r+2} + \ldots$$
$$a_{2r+1} = K_r q^{r+1}\sin 2\sigma + L_r q^{r+2}\sin 4\sigma + (M_r\sin 2\sigma + N_r\sin 6\sigma)q^{r+3} + \ldots$$

where A_r, B_r, ..., K_r, L_r, ... are purely numerical coefficients, independent of q and σ.

But if we equate to zero the coefficients of $\sin\{(2r+1)z - \sigma\}$ in the equation last used, we have

$$\{-(2r+1)^2 + \mu^2 + a\}b_{2r+1} - 2\mu(2r+1)a_{2r+1} + 8q(b_{2r-1} + b_{2r+3}) = 0$$

or

$$\{-4r(r+1) + 8q\cos 2\sigma - 8qb_3\}b_{2r+1} - 8q(2r+1)(\sin 2\sigma - a_3)a_{2r+1} + 8q(b_{2r-1} + b_{2r+3}) = 0$$

and similarly by equating to zero the coefficient of $\cos\{(2r+1)z - \sigma\}$, we have

$$\{-4r(r+1) + 8q\cos 2\sigma - 8qb_3\}a_{2r+1} + 8q(2r+1)(\sin 2\sigma - a_3)b_{2r+1} + 8q(a_{2r-1} + a_{2r+3}) = 0.$$

These are linear difference equations to determine a_{2r+1} and b_{2r+1}; they may evidently be combined into the single linear difference-equation

$$\{-4r(r+1)+8q\cos 2\sigma - 8qb_3 - 8qi(2r+1)(\sin 2\sigma - a_3)\}x_{2r+1} + 8q(x_{2r-1}+x_{2r+3})=0$$

to determine the complex coefficient $x_{2r+1}=b_{2r+1}+i\,a_{2r+1}$.

By aid of these formulae the coefficients which have been denoted by A_r, B_r, ...K_r, ... above can be determined: we find for instance that

$$A_r=\frac{2^r}{(r+1)!\,r!},\quad B_r=\frac{2^{2r+1}}{(r+1)!\,(r+1)!}$$

$$K_r=\frac{2^{r+1}}{r!\,(r+1)!}\left\{\frac{1}{r+1}+\frac{2}{r}+\frac{2}{r-1}+\frac{2}{r-2}+\ldots+\frac{2}{3}+\frac{2}{2}+1\right\}.$$

The series in the last equation can, as is well known, be summed by means of the logarithmic derivate of the Gamma-Function.

On the Quasi-periodic Solutions of Mathieu's Differential Equation.

By Andrew W. Young, Research Student in the Mathematical Laboratory, University of Edinburgh.

(*Read 13th February 1914. Received 10th June 1914*).

Mathieu's differential equation

$$\frac{d^2y}{dz^2} + (a + k^2\cos^2 z)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

is the equation which arises out of those two-dimensional problems in Mathematical Physics in which the boundary is an ellipse, such problems, for example, as the vibrations of an elliptic membrane, which was first discussed by Mathieu,* and the scattering of electromagnetic waves by a wire of elliptic cross-section. A different use of the same equation is found in Celestial Mechanics in the treatment of perturbations and oscillations about periodic orbits,† and, in a more mundane connection, it has been shown to be the differential equation of the variety artiste who holds an assistant poised on a pole above his head while he himself is standing on a spherical ball rolling on the ground!

The coefficient $a + k^2\cos^2 z$ is a periodic function of z, and hence the equation (1) belongs to the class of differential equations with periodic coefficients, a class which has been discussed by Floquet,‡ who gave what may be termed the Fuchsian theory, and, recently, by Hamel,§ who has investigated certain properties affecting the "stability" of the solutions.

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* Mathieu: Liouville's Journal (2), XIII., p. 137–203, 1868.

† Bruns: Ast. Nach. No. 2533, S. 193–204, 1883, and No. 2553, S. 129–132, 1884.

‡ Floquet: Annales de l'Ecole Normale Supérieure (2), T. 12, p. 47–88, 1883.

§ Hamel: Mathematische Annalen, B. 73, S. 371–412, 1913.

From Floquet's theory we know that the general solution of equation (1) is of the form

$$y = Ae^{cz}p_1(z) + Be^{-cz}p_2(z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where $p_1(z)$ and $p_2(z)$ are periodic functions having the same period as the coefficient of (1),

c is a constant depending on the constants of the original differential equation,

and A and B are the arbitrary constants of the solution.

Such a solution as (2) is called *quasi-periodic*, inasmuch as after the lapse of a period it does not repeat itself exactly but contains multiplying factors, thus

$$y(z + 2\pi) = e^{2\pi c}Ae^{cz}p_1(z) + e^{-2\pi c}Be^{-cz}p_2(z).$$

When the constants a and k of the original differential equation are such that $c = 0$, the above existence theorem fails to give the general solution. In this case one solution is purely-periodic, and the other solution is related to it in a way analogous to that in which, for example, the Bessel's functions of the first and second kinds are connected. These periodic solutions, which in Professor Whittaker's * notation, are written

$$ce_0(z),\ ce_1(z),\ ce_2(z),\ \ldots$$
$$se_1(z),\ se_2(z),\ \ldots$$

are infinite in number, and as Hilbert † has shown that they may be regarded as the *eigenfunktionen* (or autofunctions) of a certain integral equation, of which the corresponding *eigenwerte* (or autovalues) are the values of a concerned, we may agree to call these solutions *eigenfunktionen*, and these values of a, *eigenwerte of the differential equation*.

For values of a other than these *eigenwerte* the solution is quasi-periodic, and it is the main object of this paper to show how these quasi-periodic solutions change their nature as a is varied (k being regarded as a constant parameter), and how the quasi-periodicity

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* Whittaker: Cambridge International Congress, Vol. I., 1912.

† Hilbert: Göttingen Nachrichten, 1904, S. 213–234. The Hilbert integral equation is essentially different from that given by Whittaker (*loc. cit.*). Whittaker's has a continuous *kern*, whereas Hilbert's *kern* has a discontinuity.

merges into pure periodicity in the neighbourhood of the *eigenwerte.* With this in view we cannot make much progress by means of the method, used by Lindemann* and Maclaurin,† of transforming the equation into an algebraic equation by the substitution of $x = \cos z$ and then expanding in Taylor's series round certain points in the x-plane. Accordingly we shall confine ourselves to solutions of the form (2).

§ 1. *Convenient Form of the differential equation.*

If we write $k^2 = 32q$ in (1), with a view to saving repeated powers of 2 in the expansions, we obtain for our differential equation

$$\frac{d^2y}{dz^2} + (a + 32q\cos^2 z)y = 0$$

or

$$\frac{d^2y}{dz^2} + (a + 16q + 16q\cos 2z)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

§ 2. *Expression of the Quasi-periodic Solution in a Form which reduces to the ce_0-function.*

We know that the elliptic-cylinder function (or periodic solution) of zero order, $ce_0(z)$, is given by

$$y = ce_0(z) = 1 + 4q\cos 2z + 2q^2\cos 4z + q^3\left(\frac{4}{9}\cos 6z - 28\cos 2z\right) + \ldots$$

when $a + 16q = -32q^2 + 224q^4 - \frac{29696}{9}q^6 + \ldots$

Let us try to get a solution

$y = Ae^{cz}(\frac{1}{2} +$ a periodic function$) + Be^{-cz}(\frac{1}{2} +$ a periodic function$)$ which will reduce to $ce_0(z)$ when $c = 0$ and $A = B = 1$.

The process used is as follows:

Assume $a = a_0 + a_1q + a_2q^2 + a_3q^3 + \ldots$

and

$$y = Ae^{cz}(\tfrac{1}{2} + b_1(z)q + b_2(z)q^2 + b_3(z)q^3 + \ldots) + Be^{-cz}(\tfrac{1}{2} + d_1(z)q + d_2(z)q^2 + d_3(z)q^3 + \ldots).$$

Substituting in the equation (3), and equating to zero the terms not containing q as a factor, we have

$$\tfrac{1}{2}c^2 + \tfrac{1}{2}a_0 = 0,$$

and

$$\therefore \quad a_0 = -c^2.$$

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* Lindemann: Mathematische Annalen, B. 22, S. 117–123, 1883.

† Maclaurin: Trans. Camb. Phil. Soc., Vol. XVII., p. 41–108, 1899.

Equating to zero the terms with the first power of q as a factor, we obtain

$$b_1''(z) + 2b_1'(z)c + b_1(z)c^2 - b_1(z)c^2 + \tfrac{1}{2}a_1 + 8 + 8\cos 2z = 0,$$
$$d_1''(z) - 2d_1'(z)c + d_1(z)c^2 - d_1(z)c^2 + \tfrac{1}{2}a_1 + 8 + 8\cos 2z = 0.$$

In order that $b_1(z)$ and $d_1(z)$, the integrals of these equations may have only periodic terms, we must make

$$\tfrac{1}{2}a_1 + 8 = 0$$

and $$\therefore \quad a_1 = -16,$$

and we then have

$$b_1(z) = \frac{2\cos 2z - 2c\sin 2z}{1+c^2}.$$

$$d_1(z) = \frac{2\cos 2z + 2c\sin 2z}{1+c^2}.$$

In this way we proceed to determine in succession a_2, a_3, ..., $b_2(z)$, $b_3(z)$, ..., $d_2(z)$, $d_3(z)$, ..., and thus obtain

$$y = Ae^{cz}\left[\tfrac{1}{2} + \frac{2\cos 2z - 2c\sin 2z}{1+c^2}\cdot q + \frac{(4-2c^2)\cos 4z - 6c\sin 4z}{(1+c^2)(4+c^2)}\cdot q^2\right.$$

$$\left. + \left\{\cos 2z\left(-\frac{16(1-c^2)}{(1+c^2)^3} + \frac{8(1-2c^2)}{(1+c^2)^2(4+c^2)}\right) + \sin 2z\left(\frac{32c}{(1+c^2)^3} - \frac{4c(5-c^2)}{(1+c^2)^2(4+c^2)}\right)\right.\right.$$

$$\left.\left. + \cos 6z\,\frac{8(1-c^2)}{(1+c^2)(4+c^2)(9+c^2)} - \sin 6z\,\frac{4c(11-c^2)}{3(1+c^2)(4+c^2)(9+c^2)}\right\}q^3 + \ldots\right]$$

$$+ Be^{-cz}\,[\text{the same function of } -c]$$

when $$a + 16q = -c^2 - \frac{32}{1+c^2}\cdot q + 128\left[\frac{2(1-c^2)}{(1+c^2)^3} - \frac{(1-2c^2)}{(1+c^2)^2(4+c^2)}\right]q^3 + \ldots.$$

It will be seen that these expansions contain as denominators $(1+c^2)$, $(4+c^2)$, etc., so that they are not convergent near $c = i$, $2i$, ... Now these are the values of c which, when substituted in

$$y = Ae^{cz}\,(\text{periodic function}) + Be^{-cz}\,(\text{periodic function})$$

would give periodic solutions, and hence this expansion for the quasi-periodic solution cannot be used practically for those values of a which are in the neighbourhood of the *eigenwerte* corresponding to the ce_1, ce_2..., se_1, se_2, ... functions. We shall afterwards show that in these neighbourhoods the nature of the quasi-periodicity undergoes changes, and it is interesting to note how these changes take place under cover of the divergence of the expression, a circum-

stance in many ways analogous to the well-known property, which the constants of an asymptotic expansion have, of changing their character in the "haziness" that arises in certain regions of the argument.

§ 3. *Expression of the Quasi-periodic Solution in Forms which reduce to the Elliptic-cylinder Functions other than the ce_0-function.*

The method of the last section was used to obtain an expression which would reduce to $ce_0(z)$ for $c=0$. It is natural to try to get an expansion on similar lines which would reduce to $ce_1(z)$ or to any of the other periodic solutions, and Professor Whittaker has pointed out to me an elegant method by which we deduce an expansion reducing to ce_1 and se_1, and which can be easily extended to provide expansions reducing to any required periodic solution.*

The ce_1 solution is

$$ce_1(z) = \cos z + q\cos 3z + q^2(-\cos 3z + \tfrac{1}{3}\cos 5z) + \dots$$

when $$a + 16q = 1 - 8q - 8q^2 + 8q^3 - \dots$$

and the se_1 solution is

$$se_1(z) = \sin z + q\sin 3z + q^2(\sin 3z + \tfrac{1}{3}\sin 5z) + \dots$$

when $$a + 16q = 1 + 8q - 8q^2 - 8q^3 - \dots ;$$

and Professor Whittaker's solution shows that these are simply particular cases (corresponding to $\sigma = -\frac{\pi}{2}$ and to $\sigma = 0$) of the theorem that a quasi-periodic solution is

$$y = e^{4qz\sin 2\sigma}[\sin(z-\sigma) + q\sin(3z-\sigma) + q^2(3\sin 2\sigma\cos(3z-\sigma) + \cos 2\sigma\sin(3z-\sigma) + \tfrac{1}{3}\sin(5z-\sigma)) + \dots]$$

when $$a + 16q = 1 + 8\cos 2\sigma \,.\, q - 8(1 + 2\sin^2 2\sigma)q^2 + \dots$$

This expansion is obtained by assuming the solution

$$y = e^{Nqz\sin 2\sigma}[\sin(z-\sigma) + qb_1(z) + q^2 b_2(\sigma) + \dots]$$

along with $$a + 16q = 1 + qa_1(\sigma) + q^2 a_2(\sigma) + \dots ,$$

and finding $a_1, a_2, \dots$ in succession such that $b_1, b_2, b_3, \dots$ are periodic functions. It is found in this case that $N = 4$.

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* An account of this method is given in a note by Professor Whittaker in the present volume.

It is clear that the general solution of the equation may be written

$$y = Ae^{4qz\sin 2\sigma}[\sin(z-\sigma) + q\sin(3z-\sigma) + \ldots]$$
$$+ Be^{-4qz\sin 2\sigma}[\sin(z+\sigma) + q\sin(3z+\sigma) + \ldots]$$

when $\quad a + 16q = 1 + 8\cos 2\sigma \,.\, q - 8(1 + 2\sin^2 2\sigma) \,.\, q^2 + \ldots .$

When we aim at getting a solution which will reduce to, say, ce_n and se_n, we assume as solution

$$y = e^{Nq^n z\sin 2\sigma}[\sin(nz-\sigma) + qb_1(z) + q^2 b_2(z) + \ldots],$$

along with $\quad a + 16q = n^2 + qa_1(\sigma) + q^2 a_2(\sigma) + \ldots,$

the only change being the substitution of q^n for q in the exponential factor and in taking $\sin(nz-\sigma)$ as the solution when $q = 0$.

The case $n = 2$ gives

$$y = Ae^{-4q^2 z\sin 2\sigma}[\sin(2z-\sigma) + (2\sin\sigma + \tfrac{2}{3}\sin(4z-\sigma))q + \tfrac{1}{6}\sin(6z-\sigma) \,.\, q^2$$
$$+ \{\tfrac{8}{3}\sin\sigma - 16\sin^3\sigma - \tfrac{16}{9}\sin 2\sigma\cos(4z-\sigma)$$
$$+ (-\tfrac{5}{3} + \tfrac{16}{9}\sin^2\sigma)\sin(4z-\sigma) + \tfrac{1}{45}\sin(8z-\sigma)\}q^3 + \ldots]$$
$$+ Be^{+4q^2 z\sin 2\sigma}\,[\text{the same function of } -\sigma],$$

when $a + 16q = 4 - (\tfrac{16}{3} - 32\sin^2\sigma)q^2 + \ldots .$

This solution reduces to $ce_2(z)$ for $\sigma = -\dfrac{\pi}{2}$,

and to $se_2(z)$ for $\sigma = 0$.

The case $n = 3$ gives similarly

$$y = Ae^{\frac{4}{3}q^3 z\sin 2\sigma}[\sin(3z-\sigma) + (-\sin(z-\sigma) + \tfrac{1}{2}\sin(5z-\sigma))q$$
$$+ (-\sin(z+\sigma) + \tfrac{1}{10}\sin(7z-\sigma))q^2$$
$$+ (-\tfrac{1}{2}\sin(z-\sigma) + \tfrac{7}{40}\sin(5z-\sigma) + \tfrac{1}{90}\sin(9z-\sigma))q^3 + \ldots]$$
$$+ Be^{-\frac{4}{3}q^3 z\sin 2\sigma}\,[\text{the same function of } -\sigma]$$

when $\quad a + 16q = 9 + 4q^2 + 8\cos 2\sigma \,.\, q^3 + \ldots .$

This solution reduces to $ce_3(z)$ for $\sigma = -\dfrac{\pi}{2}$

and to $se_3(z)$ for $\sigma = 0$.

With these solutions we can explore beyond $\sigma = 0$ and $\sigma = \dfrac{\pi}{2}$ by using imaginary values of σ, but from later considerations it will be evident that any one of these solutions cannot provide an expansion for all values of the parameter a, and use will accordingly be made of these solutions only in the vicinity of the *eigenwerte* of the periodic solutions to which they naturally reduce.

§ 4. *Discussion of the variation of c.*

We shall return now to a consideration of the Floquet form of solution $y = Ae^{cz}p_1(z) + Be^{-cz}p_2(z)$, and proceed to discuss the variation of the parameter a.

When q is small—as usually happens in astronomical applications—it is evident that a first approximation to the solution of (1) is

$$y = Ae^{\sqrt{-a}.z} + Be^{-\sqrt{-a}.z},$$

so that a first approximation to the value of c is $i\sqrt{a}$.

Also if $y = Ae^{cz}p_1(z) + Be^{-cz}p_2(z)$ is a solution, we have solutions of the form

$$y = Ae^{(c\pm ni)z}p_1'(z) + Be^{-(c\pm ni)z}p_2'(z),$$

where $n = 1, 2, 3, \ldots$.

Thus c is not a single-valued quantity, since $c \pm ni$ would act equally well, but it will be convenient to select that value of c which would reduce to $i\sqrt{a}$ if q were taken to be zero. With this convention $c = 0, 1, 2, \ldots, n, \ldots$ will correspond to the periodic solutions

$$ce_0 \quad \begin{matrix} ce_1 \\ se_1 \end{matrix}\Big\} \quad \begin{matrix} ce_2 \\ se_2 \end{matrix}\Big\} \cdots \begin{matrix} ce_n \\ se_n \end{matrix}\Big\} \cdots .$$

In § 2 it is shown that, when the parameter a is not in the neighbourhood of any of the *eigenwerte* other than that of the ce_0-function, the relation between c and a is

$$a + 16q = -c^2 - \frac{32}{1+c^2}\cdot q^2 + \left(\frac{256(1-c^2)}{(1+c^2)^3} - \frac{128(1-2c^2)}{(1+c^2)^2(4+c^2)}\right)q^4 + \ldots\ldots$$

When the denominators are expanded, this becomes

$$\begin{aligned} a + 16q = &-c^2 - 32(1 - c^2 + c^4 - c^6 + \ldots)q^2 \\ &+ (224 - 888c^2 + 2046c^4 - 3712\cdot 5c^6 + \ldots)q^4 \\ &- \left(\frac{29696}{9} + \ldots\right)q^6 + \ldots \end{aligned}$$

or

$$\begin{aligned} a + 16q = &\left(-32q^2 + 224q^4 - \frac{29696}{9}q^6 + \ldots\right) \\ &- c^2(1 - 32q^2 + 888q^4 - \ldots) \\ &+ c^4(-32q^2 + 2046q^4 - \ldots) \\ &- c^6(-32q^2 + 3712\cdot 5q^4 - \ldots). \end{aligned}$$

For any particular value of q (sufficiently small to make the series in the brackets convergent), this formula gives a for any

value of c, and by a simple reversion of this equation we can derive a formula for c in terms of a.

Since general formulae have not been obtained, we must have resort to numerical calculation for further knowledge, and the result is, that when we substitute *real* values of c in the formula, values of a are obtained which are less than the *eigenwerte* of the ce_0-function, and, when we substitute *imaginary* values of c, the corresponding values of a are all found to be greater than the *eigenwerte* of ce_0.

As we noted above, this formula cannot be used near the *eigenwerte* of the other periodic solutions, and in their vicinity we must have recourse to the results of § 3.

§ 5. *Note on the Application of G. W. Hill's Method.*

In his celebrated memoir "On the Mean Motion of the Lunar Perigee," Hill* reduced the problem to the discussion of the equation

$$\frac{d^2y}{dz^2} + (\Theta_0 + 2\Theta_1\cos 2z + 2\Theta_2\cos 4z + \ldots)z = 0$$

and when $\Theta_2\Theta_3\ldots$ are taken to be zero, the equation becomes the Mathieu equation. We may thus make use of Hill's results, but the formula which he arrived at for c in terms of the Θ's suffers from the same disability as the formula of the last section, in that it cannot be used in the regions near any of the *eigenwerte*, and therefore does not give any extra information.

§ 6. *The value of c near the eigenwerte.*

The results of § 3 are shown in the table annexed, in which are arranged the values of $a + 16q$ and of c corresponding to the region of ce_1 and se_1, of ce_2 and se_2, ...

In the neighbourhood of the *eigenwerte* of	c	$a+16q$
ce_1 and se_1	$4q\sin 2\sigma$	$1 + 8\cos 2\sigma \,.\, q - 8(1 + 2\sin^2 2\sigma)q^2 + \ldots$
ce_2 and se_2	$-4q^2\sin 2\sigma$	$4 - (\frac{1}{3}\frac{6}{} - 32\sin^2\sigma)q^2 + \ldots$
ce_3 and se_3	$\frac{4}{3}q^3\sin 2\sigma$	$9 + 4q^2 + 8\cos 2\sigma \,.\, q^3 + \ldots$
.....	....	.....

* Hill: Acta Mathematica, Vol. VIII., pp. 1–36, 1886.

It will be seen that $a + 16q$ is *real* for real and imaginary values of σ, but that c is real or imaginary according as σ is real or imaginary.

Now for the real values σ, $a + 16q$ lies between the *eigenwerte* corresponding to $\sigma = 0$ and $\sigma = -\frac{\pi}{2}$ and hence we see that *c is real for values of a intermediate between eigenwerte of any pair of the periodic functions which are of the same order, e.g. ce_1 and se_1, ce_2 and se_2, etc.*

§ 7. *Variation of c over whole region of the parameter a.*

Combining the results of §§ 4 and 6 we have the following theorem :—

The values of a corresponding to the periodic solutions (or elliptic cylinder functions) serve to mark out regions in the whole range of values of a, in which the corresponding value of c is either always real or always imaginary. In particular

from $a = -\infty$ to the *eigenwerte* of ce_0, c is real ;

from the *eigenwerte* of ce_0 to the *eigenwerte* of ce_1, c is imaginary ;

,, ,, ,, ,, ce_1 ,, ,, ,, se_1, c is real ;

,, ,, ,, ,, se_1 ,, ,, ,, se_2, c is imaginary ;

,, ,, ,, ,, se_2 ,, ,, ,, ce_2, c is real ;

and so on.

It will be noticed from the last column in the table of § 6 that, since the earliest term of the series containing a function of σ is that in q^n, the interval between the *eigenwerte* belonging to the ce_n-function, and that belonging to the se_n-function depends on q^2, and hence, if q be small, decreases rapidly with increasing values of the order n of the periodic solutions. From the second column c is likewise seen to depend on q^n.

§ 8. *Stability and Instability of the Solutions.*

In most of the physical applications of Mathieu's differential equation it is the periodic solution that is wanted, but in the astronomical questions it is the quasi-periodic, and, more especially, it is the factor of quasi-periodicity, c, that is of importance.

The equation

$$\frac{d^2y}{dz^2} + (a + 16q + 16q\cos 2z)y = 0$$

may be regarded as specifying the effect of perturbative forces on a system which is moving in a periodic orbit, y being the variable expressing the variation from the periodic trajectory and z being the measure of the time. To each pair of values of a and q there will be an orbit from which the differential equation expresses the variation, and from the nature of the solution

$$y = Ae^{cz}p_1(z) + Be^{-cz}p_2(z)$$

we see that the periodic orbit corresponding to any pair of values of a and q will be stable or unstable according as c is imaginary or real. Confining our attention to one fixed value of q, we have an infinite family of periodic orbits corresponding to all possible values of a, and the results of last paragraph show how the instability or stability can be determined and how the property changes at each periodic solution of the equation of perturbation. The constant c may be regarded as giving the period of the small oscillation from the periodic orbit which is caused by the disturbing force; this period of small oscillation will gradually change as we proceed from one periodic orbit to another until we come to one for which the period of small oscillation is an aliquot part of the period of the orbit. Beyond this particular orbit there is instability until we arrive at another coincidence of the periods of the small oscillations and the ordinary orbit.

On the Attractions of Spherical and Ellipsoidal Shells.

By Professor A. Gray, F.R.S.

(*Received 1st June 1914. Read 13th June 1914.*)

So far as I am aware, the methods, described below, of finding the attraction of a uniform shell of matter on a particle placed at an external or an internal point, are new. They are particular applications of a method of finding the attraction of a thin shell of uniform volume density bounded by similar and similarly situated ellipsoidal surfaces (what some have called a *homothetic shell*, and Thomson and Tait an *elliptic homœoid*) which I explained in a paper, on the attraction of ellipsoidal shells and of solid ellipsoids, which was published in the *Philosophical Magazine* for April 1907. I give here also a short account of the more general problem with some additional notes and remarks. The solution depends on a geometrical theorem of some interest which occurred to me in thinking over the problem of the ellipsoidal shell, and its solution by Poisson by a laborious and somewhat difficult process of integration (*Mémoires de l'Institut*, t. xv., 1835).

1. Consider a thin uniform spherical shell, of radius a and mass σ per unit of area, attracting a particle of unit mass at a point P distant f from the centre C of the shell. (1) Let P be external to the shell as shown in Fig. 1. An element of area dS at E has mass σdS, and its attraction on the unit particle at P is $\kappa\sigma d\text{S}/r^2$, where r denotes the distance EP, and κ the gravitation constant. Resolving along the direction of the resultant attraction of the shell on the particle, putting θ for the angle APE, we obtain for the attraction F of the whole shell the equation

$$\text{F} = \kappa\sigma\int\frac{\cos\theta}{r^2}\,d\text{S}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where the integral is taken over the surface of the shell. Now,

describing through P a surface concentric with the shell, producing CE to meet that surface in E′ and joining AE′, we see that $AE' = r$, and $\angle AE'E = \theta$. Moreover, by radial projection of dS from the centre C, we obtain an area dS' on the concentric surface such that $dS = dS' \cdot a^2/f^2$. Replacing dS in equation (1) by this value we get, integrating over the concentric surface,

$$F = \kappa\sigma \frac{a^2}{f^2} \int \frac{\cos\theta}{r^2}\, dS' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

But, since $AE' = r$ and $\angle AE'E = \theta$, $dS'\cos\theta/r^2$ is the solid angle subtended at A by the element of surface dS'. The integral is

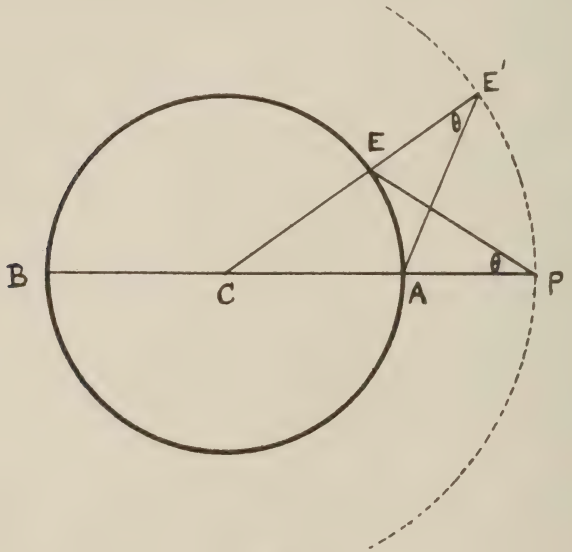

Fig. 1.

therefore the solid angle subtended by the whole concentric surface at the internal point A, or 4π. Hence

$$F = \kappa \frac{4\pi\sigma a^2}{f^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

that is, the attraction is the same as if the whole mass were collected at C.

(2) Let P be an internal point, as shown in Fig. 2, and a concentric surface be described through P as before. Corre-

sponding to dS there is an element of surface dS' as before on the concentric surface, and again $AE' = PE = r$, $\angle APE = \angle AE'E = \theta$. Thus, again,

$$F = \kappa \frac{a^2}{f'^2} \int \sigma dS' \frac{\cos\theta}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Now, however, the integral is the solid angle subtended by the concentric surface at the *external* point A. This, of course, is zero, and therefore $F = 0$.

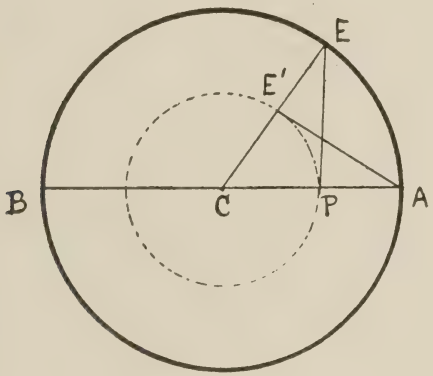

Fig. 2.

It will be noticed that, if the shell be incomplete, the attraction is in each case the product of $\kappa\sigma a^2/f^2$ by the proper value of the solid angle subtended at A by the part of the concentric surface corresponding to the incomplete shell. For example, let P be external to the shell, regarded as complete. It is obvious that if we divide the shell into two segments, by drawing the polar plane of P with respect to the shell, the solid angles subtended at A by the corresponding parts of the concentric surface are each 2π. Hence the shell is divided by the plane at right angles to CP drawn through the point P', the inverse of P with respect to the shell, into two segments attracting equally the unit particle at P.

Of course this last result is obvious from another point of view. We have only to draw narrow cones through P so as to intersect the shell. The two elements intercepted by each cone have equal attractions on the particle at P. The elements fall into two sets,

one set forming one of the two segments just specified the other set forming the other segment.

From the result we can easily find a solution of the problem of dividing a solid sphere into two parts which shall have equal attractions at any point P. Let the centre be C: on CP as diameter describe a sphere. This spherical surface divides the solid sphere into two parts which have equal attractions on a particle at P. These parts are lens-shaped, one, the farther from P, convexo-concave, the other convexo-convex. The spherical surface divides each of the concentric shells, of which the solid sphere may be regarded as built up, into two segments which join along the polar plane of P with respect to the shell.

This theorem holds whether the solid sphere be of uniform density or of density varying as a function of the distance from the centre. It also holds whether P be internal or external to the sphere. For the case of uniform density the theorem can be verified in this case at once, for the attractions of uniform spheres of different radii, but of the same density, on a unit particle in contact with the surface in each case, are proportional to the radii.

2. We now pass to the more general theorem. It was proved by Poisson, in the memoir referred to above, that the resultant attraction of an elliptic homœoid at an external point is directed along the internal axis, PQ, of the cone drawn from the external point as vertex to envelop the homœoid. The full significance of his theorem does not seem to have been perceived by Poisson. For that axis of the cone is the normal to the confocal ellipsoidal surface drawn through the external point, and the theorem shows that the family of external confocal ellipsoidal surfaces are the equi-potential surfaces of the homœoid, a fact of great importance in view of Green's theory of equivalent distribution. [This was contained in his celebrated *Essay* which was published privately in 1828, but remained unknown to the French mathematicians, and to the scientific world generally, until about twenty years later, when it was republished in Crelle's *Journal*.]

Refer to Fig. 3, which represents a section of the ellipsoidal shell, the attraction of which on a unit particle at P is to be considered, by a plane containing the internal axis PQ of the enveloping cone drawn from P as vertex. The confocal surface

through P is indicated, and to this surface the axis in question is a normal. The equation of the confocal is

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=k\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and the equations of the outer and inner surfaces of the homœoid are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k-dk, \ldots\ldots\ldots\ldots (6)$$

respectively.

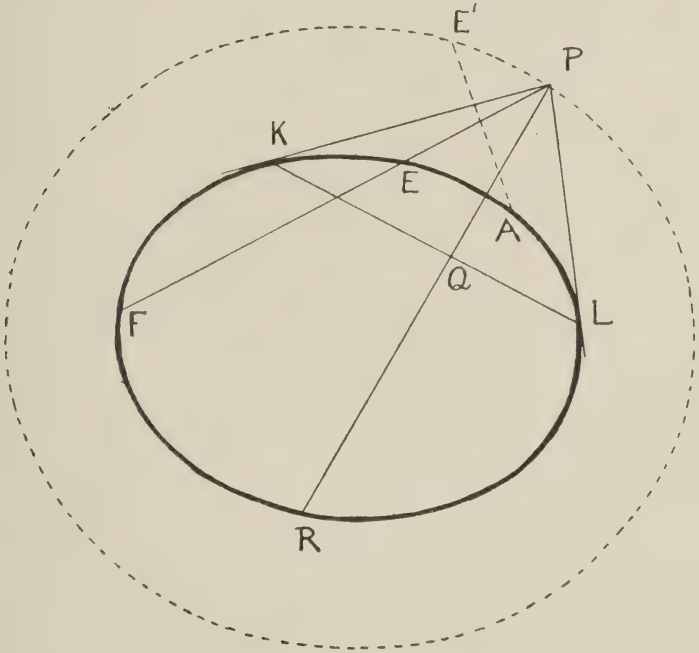

Fig. 3.

Imagine drawn from P as vertex a narrow cone of solid angle $d\omega$, intercepting two elements of the homœoid at E and F, and let the area of the element at E be dS. If the length of the perpendicular from the centre on the tangent plane at E be p, the mass of the element is $\frac{1}{2}p\rho dk d\mathrm{S}/k$. Now to dS we have by the

theory of corresponding points on confocal surfaces a corresponding element at E′, the area of which, $d\Sigma$, is given by the equation

$$pd\mathrm{S} = \frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\varpi d\Sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

where ϖ is the length of the perpendicular from the centre on the tangent plane to the confocal at E′.

3. Before proceeding further it is necessary to explain and indicate a proof of the geometrical theorem by which the attraction integral is to be transformed. Let the point A on the surface of the shell correspond to P on the confocal. The point E corresponds to E′, and by the property of corresponding points AE′ = EP, that is, AE′ is also equal to r. Let ϖ_0 be the length of the perpendicular from the centre on the tangent plane to the confocal at P, and θ be the angle between the perpendicular (of length ϖ) from the centre on the tangent plane at E′ and the line AE′. Let the angle between the former perpendicular and the line EP be θ_0. [It is important to take the lines EP, AE′ in the senses indicated by the letters, and to take the perpendiculars both outward or both inward, to avoid any difficulty as to the signs of the cosines]. Then it can be proved that

$$\varpi\cos\theta_0 = \varpi_0\cos\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

This is the geometrical theorem employed. I have not seen it referred to in works on geometry.

Of course the theorem also holds when the perpendiculars are those to the tangent planes which touch at the two points A, E on the shell. The proof in either case is easy: it is only necessary to express ϖ_0, ϖ, $\cos\theta_0$, $\cos\theta$, by means of the equation of the confocal and the coordinates of P and E′ when it is found that (8) is verified. The theorem of course is not confined to the particular case of *ellipsoids*.

Since an ellipsoid is confocal with itself, the theorem also holds when P coincides with A and E′ with E. Then EP coincides with EA and AE′ with AE. Hence if p_0, p be the lengths of the perpendiculars from the centre to the tangent planes at the extremities of a chord drawn to join any two points A, E on an ellipsoid, and

θ_0, θ be the angles between AE and p_0 and between EA and p respectively, we have again

$$p\cos\theta_0 = p_0\cos\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8')$$

4. This latter case of the geometrical theorem enables us to prove Poisson's theorem of the direction of the resultant attraction, and to establish some other results. From P draw a narrow cone, of solid angle $d\omega$, intercepting elements of area $d\mathrm{S}_1, d\mathrm{S}_2$ at E, F on the shell, and let p_1, p_2 be the lengths of the perpendiculars from the centre on the tangent planes at E, F, θ_1, θ_2 the angles between these perpendiculars and EF, FE respectively. Then clearly we have $d\mathrm{S}_1 = r_1^2 d\omega/\cos\theta_1$, $d\mathrm{S}_2 = r_2^2 d\omega/\cos\theta_2$, where $r_1 = \mathrm{PE}$, $r_2 = \mathrm{PF}$. But by the statement made above, § 2, as to the mass of an element of the homœoid, the masses intercepted at E and F are

$$\tfrac{1}{2}p_1\rho dk d\mathrm{S}_1/k, \quad \tfrac{1}{2}p_2\rho dk d\mathrm{S}_2/k$$

which, by the values just found for $d\mathrm{S}_1$, $d\mathrm{S}_2$ are

$$\tfrac{1}{2}\rho dk r_1^2 d\omega p_1/k\cos\theta_1, \quad \tfrac{1}{2}\rho dk r_2^2 d\omega p_2/k\cos\theta_2.$$

The attractions on a particle of unit mass at P are therefore

$$\tfrac{1}{2}\rho dk d\omega p_1/k\cos\theta_1, \quad \tfrac{1}{2}\rho dk d\omega p_2/k\cos\theta_2,$$

and these are equal by (8′). It is clear, therefore, that the attracting elements fall into two sets, one set on the side of the polar plane towards P the other set on the side remote from P. The polar plane therefore divides the homœoid into two segments which exert equal attractions on a particle at P.

Moreover, for any narrow cone taken with its axis on one side of the axis of the enveloping cone, another cone of equal solid angle can be taken on the other side of the latter axis in the same plane with it, and equally inclined to it. By such pairs of cones the homœoids can be exactly exhausted, and so Poisson's theorem of the direction of the attraction at P is established.

A similar method to that just employed for the point P can be used to show that the attraction on a particle at any internal point is zero. It is only necessary to draw cones with their vertices at the point chosen, and apply the theorem (8′) as before. All reference to the homœoid as found by pure strain from a uniform spherical shell is thus rendered unnecessary. We shall obtain presently another proof of this theorem.

A proof of Poisson's theorem of the direction of the attraction was given by Steiner (*Crelle*, Bd. 12, 1834) after the *announcement* of the theorem in 1833. This, however, involved the assumption, to be justified by derivation of the homœoid from the spherical shell, that two elements intercepted by the same narrow cone, have masses in the ratio of the squares of their distances from the vertex of the cone. Steiner's proof is reproduced in the *Phil. Mag.* paper (*loc. cit.*).

5. We can now find the magnitude F of the resultant. We have, if r denote the distance EP, $d\mathrm{S}$ the area of an element at E, p the length of the perpendicular from the centre on the tangent plane at E, and θ_0 the angle EPQ,

$$\mathrm{F} = \tfrac{1}{2}\kappa\rho\frac{dk}{k}\int p\cos\theta_0\frac{d\mathrm{S}}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

The integration is taken over the homœoid. The difficulty in Poisson's investigation was the evaluation of this integral, and its accomplishment was a rather troublesome and lengthy process.

If in (9) we make the substitution given by (7), we get

$$\mathrm{F} = \tfrac{1}{2}\kappa\rho\frac{dk}{k}\ \frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\int\varpi\cos\theta_0\frac{d\Sigma}{r^2} \quad \ldots\ldots\ldots(10),$$

or by the relation (8), since ϖ_0 is the same for every element,

$$\mathrm{F} = \tfrac{1}{2}\kappa\rho\frac{dk}{k}\ \frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\varpi_0\int\frac{\cos\theta d\Sigma}{r^2}. \quad \ldots\ldots\ldots(11)$$

In (10) and (11) the integrations are to be taken over the confocal surface. Clearly the integral in (11) is the solid angle subtended at the point A by the surrounding confocal surface, and is therefore 4π. Hence we have

$$\mathrm{F} = 2\pi\kappa\rho\varpi_0\frac{dk}{k}\ \frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \quad \ldots\ldots\ldots\ldots(12)$$

which is equivalent to Poisson's result. From this (see *Phil. Mag.*, *loc. cit. supra*) the potential of the homœoid at P, and at any other point is easily obtained.

We can proceed in exactly the same way when P is within the homœoid, by taking a confocal ellipsoidal surface through P. We have then for the equation of the confocal

$$\frac{x^2}{a^2-u}+\frac{y^2}{b^2-u}+\frac{z^2}{c^2-u}=k,$$

and get by the same process the same equation (12), except that u is replaced by $-u$. Here, however, the integral is the solid angle subtended by the closed surface of the confocal at the *external* point A, and is zero. Hence F is zero at P.

As in the case of the spherical shell, if the homœoid is incomplete its attraction is given by estimation of the corresponding solid angle subtended by the confocal at A.

Also, it can easily be shown that the parts of the confocal corresponding to the two segments, into which the shell is divided by the polar plane, subtend each an angle 2π at the point A, so that these segments exert equal attractions at the external point P, as proved more simply above.

6. It is clear that the surface of which the equation is

$$\frac{x^2-fx}{a^2}+\frac{y^2-gy}{b^2}+\frac{z^2-hz}{c^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

contains the intersections of all the polar planes of point P (coordinates f, g, h) with respect to the surfaces (common centre C) obtained by giving different values to k in the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Hence if f, g, h be the coordinates of the point P at which the unit particle is situated, and an ellipsoid, of which the equation (referred to the same axes as are used for the homœoids) is (13), be described from the point of coordinates $(\frac{1}{2}f, \frac{1}{2}g, \frac{1}{2}h)$ as centre, it will contain the intersections with the surfaces of all the polar planes of the point P with respect to the family of surfaces represented by (14). Thus it will divide each homœoid characterised by an assigned value of k into two segments which exert equal attractions on a particle at P.

Further, if a solid ellipsoid be constructed by placing together concentric homœoidal shells, of which the equations of the succes-

sive surfaces are given by varying k, and which have each uniform volume density, varying in any manner from shell to shell, the ellipsoidal surface of which the equation is (13) will divide the solid ellipsoid into two parts which attract equally a unit particle at P. This holds even when P is within the solid ellipsoid. The centre of this surface is the mid-point of CP, and obviously CP is a diameter.

Thus the results stated in the first part of this paper for spherical distributions have their exact counterparts in the general case of ellipsoidal distributions.

On certain theorems in Continued Fractions equivalent to Riemann's and other Transformations of the P-function.

By E. Lindsay Ince.

(*Read 8th May 1914. Received 21st May 1914*).

§ 1. *Introduction.*

The transformation of Continued Fractions into one another is a subject in which very little work has so far been done. Beyond the simple transformations given in works on elementary algebra, few transformation-theorems are known; the best known being those connected with the "contraction" and "extension" of Continued Fractions, and the transformations of Euler, Bauer and Muir.

The purpose of the following paper is to add to the number of known transformations. The method used depends on the fact that every linear differential equation of the second order leads to a continued fraction, so that theorems involving the transformation of one such differential equation into another would be expected to lead to transformations of continued fractions hitherto unknown.

§ 2. *The General Method.*

Let $$\frac{d^2y}{dz^2}+p(z)\frac{dy}{dz}+q(z)y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be a linear differential equation of the second order; then if we differentiate it with respect to the independent variable z we obtain a linear differential equation of the third order. We may eliminate y between equation (1) and this new equation, and thus obtain an equation of the form

$$\frac{d^3y}{dz^3}+p_1(z)\frac{d^2y}{dz^2}+q_1(z)\frac{dy}{dz}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From this equation we obtain by differentiating with respect to z and eliminating $\frac{dy}{dz}$ a third equation

$$\frac{d^4y}{dz^4}+p_2(z)\frac{d^3y}{dz^3}+q_2(z)\frac{d^2y}{dz^2}=0.$$

We may continue this process indefinitely; after n successive differentiations and the corresponding eliminations we obtain the equation

$$\frac{d^{n+2}y}{dz^{n+2}} + p_n(z)\frac{d^{n+1}y}{dz^{n+1}} + q_n(z)\frac{d^n y}{dz^n} = 0.$$

Now, denoting $\frac{dy}{dz}$ by y'; $\frac{d^2y}{dz^2}$ by y'', and so on, we obtain from equation (1) the relation

$$\frac{y'}{y} = -\frac{q}{p + \frac{y''}{y'}}.$$

But $\frac{y''}{y'}$ may be obtained from equation (2) in a similar form, viz.

$$\frac{y''}{y'} = \frac{-q_1}{p_1 + \frac{y'''}{y''}},$$

and thus we get the relation

$$\frac{y'}{y} = \frac{-q}{p - \frac{q_1}{p_1 + \frac{y'''}{y''}}}.$$

Continuing this process, and making use of the ratios of successive differential coefficients of y deduced from the equation already obtained, we finally get the ratio $\frac{y'}{y}$ expressed in terms of a continued fraction, viz.

$$\frac{y'}{y} = -\frac{q}{p} - \frac{q_1}{p_1} - \frac{q_2}{p_2} - \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Next suppose that we apply the transformation

$$\left.\begin{aligned} y &= v\phi(x) \\ z &= \psi(x) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

to the differential equation (1), and that it is changed thereby into the new equation

$$\frac{d^2v}{dx^2} + \mathrm{P}(x)\frac{dv}{dx} + \mathrm{Q}(x)v = 0 \ldots\ldots\ldots\ldots\ldots (5)$$

From this new equation (5), by applying the process above described, we obtain $\frac{v'}{v}$ in terms of a continued fraction, thus:

$$\frac{v'}{v} = -\frac{Q}{P} - \frac{Q_1}{P_1} - \frac{Q_2}{P_2} - \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

when the Q's correspond to the q's of the original equation and those derived from it by the process of differentiation and elimination, and the P's in like manner to the p's.

Thus the transformation (4) converts the continued fraction of equation (3) into that of equation (6), always provided, however, that v is the same solution of equation (5) as y is of equation (1). Thus if we denote the first continued fraction by F, we have, applying the transformation (4),

$$F = \frac{1}{y}\frac{dy}{dz} = \frac{1}{v\phi}\left(\frac{v'\phi + v\phi'}{\psi'}\right)$$
$$= \frac{1}{\psi'}\left(\frac{v'}{v} + \frac{\phi'}{\phi}\right),$$

dashes denoting differentiation with respect to the new independent variable x.

Now $\frac{v'}{v}$ is the continued fraction obtained from the transformed differential equation (5) and may be denoted by G.

Then we have the relation

$$F = \frac{1}{\psi'}\left(G + \frac{\phi'}{\phi}\right)$$

or

$$G = F\psi' - \frac{\phi'}{\phi}. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The question as to whether v and y are or are not the same solution of the transformed and the original differential equation will be discussed subsequently in the different cases.

§ 3. *Application to equations of hypergeometric type.*

In general, however, the successive differential equations become more and more involved, so that it becomes extremely difficult to obtain a general expression for the n^{th} derived equation.

Nevertheless in certain simple cases we may proceed without difficulty to the n^{th} derived equation and obtain the corresponding continued fraction.

Such a case arises in connection with the differential equation

$$(az^2+bz+c)\frac{d^2y}{dz^2}+(ez+f)\frac{dy}{dz}+gy=0, \quad \ldots\ldots\ldots\ldots\ldots (8)$$

where a, b, c, e, f and g are constants.

If we differentiate this with respect to the independent variable z we obtain the equation

$$(az^2+bz+c)\frac{d^3y}{dz^3}+\{(2a+e)z+(b+f)\}\frac{d^2y}{dz^2}+(e+g)\frac{dy}{dz}=0,$$

in which the dependent variable y does not appear and no elimination is necessary.

By differentiating equation (8) n times we finally arrive at the equation

$$(az^2+bz+c)\frac{d^{n+2}y}{dz^{n+2}}+\{(2na+e)z+(nb+f)\}\frac{d^{n+1}y}{dz^{n+1}}$$
$$+\{n(n-1)a+ne+g\}\frac{d^ny}{dz^n}=0. \quad \ldots\ldots\ldots\ldots (9)$$

These equations lead directly to the relation

$$F=\frac{y'}{y}=-\cfrac{\frac{g}{d}}{\cfrac{ez+f}{d}}-\cfrac{\frac{e+g}{d}}{\cfrac{(2a+e)z+(b+f)}{d}}-\cfrac{\frac{2a+2e+g}{d}}{\cfrac{(4a+3)z+(2b+f)}{d}}-\ldots-\cfrac{\frac{n(n-1)a+ne+g}{d}}{\cfrac{(2na+e)z+(nb+f)}{d}}-$$

$$=\cfrac{-g}{ez+f}-\cfrac{(e+g)d}{(2a+e)z+(b+f)}-\cfrac{(2a+2e+g)d}{(4a+e)z+(2b+f)}-\ldots-\cfrac{\{n(n-1)a+ne+g\}d}{(2na+e)z+(nb+f)}-\ldots..(1$$

where $d\equiv az^2+bz+c$.

In order that this continued fraction may converge and thus be of any practical utility, it is necessary that in the equation

$$(az^2+bz+c)\frac{d^2y}{dz^2}+(ez+f)\frac{dy}{dz}+gy=0$$

the first term $az^2+bz+c\equiv d$ should be of less importance than the other two, so that a good approximation is obtained by discarding it, in which case a first approximation to its value is

$$\frac{1}{y}\frac{dy}{dz}=-\frac{g}{ez+f}.$$

Thus in the case of the ordinary hypergeometric equation

$$z(z-1)\frac{d^2y}{dz^2}+\{-c+(a+b+1)z\}\frac{dy}{dz}+aby=0, \ldots\ldots..(11)$$

where z is small (*i.e.* near the singularity 0), the two last terms are non-zero, but the first term tends to zero when z tends to zero, so that here the condition is satisfied.

§ 4. *Nature of the solution y in the preceding section.*

In order to determine which solution of the differential equation (8) y happens to be, we may regard the matter from the point of view of the Riemann P-function, which is defined as being any solution of the generalised hypergeometric equation

$$\frac{d^2y}{dz^2}+\left\{\frac{1-\alpha-\alpha'}{z-a}+\frac{1-\beta-\beta'}{z-b}+\frac{1-\gamma-\gamma'}{z-c}\right\}\frac{dy}{dz}$$

$$+\left\{\frac{\alpha\alpha'(a-b)(a-c)}{z-a}+\frac{\beta\beta'(b-c)(b-a)}{z-b}+\frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right\}$$

$$\frac{y}{(z-a)(z-b)(z-c)}=0 \ldots\ldots (12)$$

and is denoted by

$$\mathrm{P}\left\{\begin{array}{cccc} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\}.$$

The generalised hypergeometric equation has three regular singular points a, b, c, with exponents α and α' at a, β and β' at b, and γ and γ' at c, respectively.

The P-function corresponding to the differential equation (8) is of the form

$$\mathrm{P}\left\{\begin{array}{cccc} \lambda & \infty & \mu & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\}$$

where λ and μ are the roots of the equation

$$az^2+bz+c=0. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Near the value λ of z, there will be two distinct expansions of y.

One, viz. y_1, is

$$y_1=(z-\lambda)^{\alpha}+\mathrm{A}(z-\lambda)^{\alpha+1}+\ldots$$

whence $$y_1'=\alpha(z-\lambda)^{\alpha-1}+(\alpha+1)\mathrm{A}(z-\lambda)^{\alpha}+\ldots$$

and $$\frac{y_1'}{y_1}=\frac{\alpha}{z-\lambda}+\mathrm{K}+\mathrm{L}(z-\lambda)+\mathrm{M}(z-\lambda)^2+\mathrm{N}(z-\lambda)^3+\ldots$$

For the other, y_2, we shall have

$$\frac{y_2'}{y_2} = \frac{\alpha'}{z-\lambda} + \mathrm{K}' + \mathrm{L}'(z-\lambda) + \dots .$$

The other notable solutions are as follows :

At infinity,

$$y_3 = \frac{1}{z^\beta} + \frac{\mathrm{A}}{z^{\beta+1}} + \frac{\mathrm{B}}{z^{\beta+2}} + \dots .$$

For this

$$y_3' = -\frac{\beta}{z^{\beta+1}} + \dots$$

and

$$\frac{y_3'}{y_3} = -\frac{\beta}{z} + \frac{\mathrm{K}''}{z^2} + \frac{\mathrm{L}''}{z^3} + \frac{\mathrm{M}''}{z^4} + \dots,$$

so also

$$\frac{y_4'}{y_4} = -\frac{\beta'}{z} + \frac{\mathrm{K}'''}{z^2} + \frac{\mathrm{L}'''}{z^3} + \frac{\mathrm{M}'''}{z^4} + \dots .$$

Near $z=\mu$, we have, in like manner,

$$\frac{y_5'}{y_5} = \frac{\gamma}{z-c} + \dots$$

and

$$\frac{y_6'}{y_6} = \frac{\gamma'}{z-c} + \dots .$$

Our purpose is now to find out which of these ratios, viz.,

$$\frac{y_1'}{y_1}, \quad \frac{y_2'}{y_2} \quad \dots \quad \frac{y_6'}{y_6},$$

is represented by the continued fraction we have obtained.

Now any equation of the form (8) may be reduced to the form of the ordinary hypergeometric equation,

$$z(z-1)\frac{d^2y}{dz^2} + \{-\gamma + (\alpha+\beta+1)z\}\frac{dy}{dz} + \alpha\beta y = 0 \quad \dots\dots\dots\dots\dots(11)$$

by moving the singularities to the points $z=0$ and $z=1$, and making zero one of the exponents α and α' and one of γ and γ'.

The corresponding P-function is

$$\mathrm{P}\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ 0 & \alpha & 0 & z \\ 1-\gamma & \beta & \gamma-\alpha-\beta & \end{array}\right\}$$

and the six notable solutions and the corresponding ratios $\frac{y'}{y}$ are

i. $y_1 = 1 + \frac{\alpha\beta}{\gamma}z + \ldots\ldots\ldots\ldots$ $\quad \frac{y_1'}{y_1} = \frac{\alpha\beta}{\gamma} + \ldots\ldots\ldots\ldots\ldots$

ii. $y_2 = z^{1-\gamma} + Az^{-\gamma} + \ldots\ldots\ldots$ $\quad \frac{y_2'}{y_2} = \frac{1-\gamma}{z} + \ldots\ldots\ldots\ldots\ldots$

iii. $y_3 = \frac{1}{z^{\alpha}} + \frac{A_1}{z^{\alpha+1}} + \ldots\ldots\ldots$ $\quad \frac{y_3'}{y_3} = -\frac{\alpha}{z} + \ldots\ldots\ldots\ldots\ldots$

iv. $y_4 = \frac{1}{z^{\beta}} + \frac{A_2}{z^{\beta+1}} + \ldots\ldots\ldots$ $\quad \frac{y_4'}{y_4} = -\frac{\beta}{z} + \ldots\ldots\ldots\ldots\ldots$

v. $y_5 = 1 + \frac{\alpha\beta}{\gamma-\alpha-\beta-1}(z-1) + \ldots$ $\quad \frac{y_5'}{y_5} = \frac{\alpha\beta}{\gamma-\alpha-\beta-1} + \ldots\ldots\ldots\ldots$

vi. $y_6 = (z-1)^{\gamma-\alpha-\beta} + \ldots\ldots\ldots$ $\quad \frac{y_6'}{y_6} = \frac{\gamma-\alpha-\beta}{z-1} + \ldots\ldots\ldots\ldots\ldots$

Of these six solutions, the solution y_1 corresponds to the solution $y = F(\alpha, \beta, \gamma, z)$ of the hypergeometric equation, where $F(\alpha, \beta, \gamma, z)$ is the hypergeometric series $1 + \frac{\alpha\beta}{1.\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}z^2 + \ldots$

Now a continued fraction corresponding to the hypergeometric equation (11) is

$$\frac{y'}{y} = \frac{-\alpha\beta}{(\alpha+\beta+1)z-\gamma} \; \frac{(\alpha+1)(\beta+1)z(z-1)}{-(\alpha+\beta+3)z-(\gamma+1)} - \ldots - \frac{(\alpha+n)(\beta+n)z(z-1)}{-(\alpha+\beta+2n+1)z-(\gamma+n)} - \ldots\ldots\ldots(14)$$

But when z is zero this reduces to $\frac{\alpha\beta}{\gamma}$, and consequently this continued fraction is no other than $\frac{y_1'}{y_1}$ where y_1 is the solution corresponding to the exponent 0 at the singularity 0. When $z = 1$, it reduces to $\frac{\alpha\beta}{\gamma-\alpha-\beta-1}$, and hence represents $\frac{y'_5}{y_5}$, thus also corresponding to the exponent 0 at the singularity 1.

§ 5. *Possible Transformations.*

The problem now arises as to what transformations are applicable, *i.e.* what transformation would change equation (8) into another equation from which a continued fraction could be completely determined.

Obviously any transformation

$$y = v\phi(x)$$
$$z = \psi(x)$$

which changed equation (8) into an equation of the form

$$(Ax^2 + Bx + C)\frac{d^2v}{dx^2} + (Ex + F)\frac{dv}{dx} + Gv = 0$$

would satisfy these conditions, provided always that the solution of this latter equation involved in the continued fraction $\frac{v'}{v}$ is that solution which corresponds to the solution of the former equation (8) involved in the continued fraction $\frac{y'}{y}$.

To obtain such a transformation, we again have recourse to Riemann's P-function. The P-function corresponding to equation (8) is

$$P\left\{\begin{array}{cccc} \lambda & \infty & \mu & \\ 0 & \beta & 0 & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\}$$

where

$$\alpha' = 1 + \frac{e\lambda + f}{a(\mu - \lambda)}$$

$$\gamma' = 1 + \frac{e\mu + f}{a(\lambda - \mu)}$$

$$2\beta = \frac{e}{a} - 1 + \frac{\sqrt{e^2 + a^2 - 2ae - 4ag}}{a}$$

$$2\beta' = \frac{e}{a} - 1 - \frac{\sqrt{e^2 + a^2 - 2ae - 4ag}}{a}$$

and λ and μ are as already defined.

Thus any such transformation which changes

$$P\left\{\begin{array}{cccc} \lambda & \infty & \mu & \\ 0 & \beta & 0 & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\}$$

into

$$P\left\{\begin{array}{cccc} \Lambda & \infty & M & \\ 0 & B & 0 & x \\ A' & B' & \Gamma' & \end{array}\right\}$$

is suited to our purpose. Several such transformations are known to exist, and examples of their application are given in the following sections.

§ 6. *Riemann's Transformation.*

It is, of course, not necessary to know the transformation itself, provided we are given a relation between two P-functions, for we can construct directly the differential equation corresponding to each of them and at once proceed to the relation between the corresponding continued fractions.

One such relation between two P-functions is given by Riemann, viz.

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \beta & 0 & z^2 \\ \frac{1}{2} & \beta' & \gamma' & \end{matrix}\right\} = P\left\{\begin{matrix} -1 & \infty & 1 & \\ 0 & 2\beta & 0 & z \\ \gamma' & 2\beta' & \gamma' & \end{matrix}\right\} \quad \ldots\ldots(15)$$

The P-function on the left hand closely resembles that corresponding to the ordinary hypergeometric equation

$$x(x-1)\frac{d^2y}{dx^2} + \{-c + (a+b+1)x\}\frac{dy}{dx} + aby = 0$$

provided that for x we write z^2 and for c we substitute the particular value $\frac{1}{2}$.

Then if $\beta = a$, $\beta' = b$, and $\gamma' = \frac{1}{2} - a - b$ the relation becomes

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & x \\ \frac{1}{2} & b & \frac{1}{2}-a-b & \end{matrix}\right\} = P\left\{\begin{matrix} -1 & \infty & 1 & \\ 0 & 2a & 0 & z \\ \frac{1}{2}-a-b & 2b & \frac{1}{2}-a-b & \end{matrix}\right\} \quad (15')$$

By comparing the right hand member of this equation with the P-function of equation (8) we easily find that it corresponds to the differential equation

$$(z^2 - 1)\frac{d^2y}{dz^2} + (2a + 2b + 1)z\frac{dy}{dz} + 4aby = 0.$$

From this we obtain the continued fraction

$$\frac{1}{y}\frac{dy}{dz} = \frac{-4ab}{(2a+2b+1)z -} \frac{(2a+1)(2b+1)(z^2-1)}{(2a+2b+3)z} - \ldots - \frac{(2a+n)(2b+n)(z^2-1)}{(2a+2b+2n+1)z} - \ldots\ldots\ldots(16)$$

It appears from this expression, and from the differential equation from which it is derived, that this C.F. is not convergent near the origin. In fact, from the corresponding P-function we see that the origin is not a singularity in the case, whereas in the left hand P-function, the origin is a singular point.

Now the continued fraction we derived in a previous section from the hypergeometric equation was shown to correspond to the solution $y = F(a, b, c, z)$ which is an expansion in the domain of the

singularity $z=0$, and thus these two continued fractions may not be equivalent in this particular region.

We must therefore find out which of the other five distinct solutions of the hypergeometric equation really is equivalent to the continued fraction (16). This continued fraction is obviously convergent in the domains of the singularities $z=1$ and $z=-1$, so that the appropriate solution of the hypergeometric equation is one which is convergent near one or other of these singular points.

The six distinct solutions of the hypergeometric equation may be denoted by $Y_1, Y_2, \ldots Y_6$, which is the notation adopted by Forsyth (Differential Equations, § 123). Of these Y_1 and Y_2 are expansions in the domain of $z-0$ with exponents 0 and $1-c$ respectively; Y_5 and Y_6 are expansions at infinity with exponents a and b; all these are to be excluded. The expansions near the singularity $z=1$ are Y_3 and Y_4, the exponents being 0 and $c-a-b$ respectively.

Our continued fraction (16) corresponds to an expansion with exponent 0 at the singularity $z=1$, so that of the solutions of the hypergeometric equation Y_3 is to be taken.

A typical member of the class Y_3 is

$$y_5 = F(a,\ b,\ a+b-c+1,\ 1-x),$$

which gives

$$-\frac{1}{y_5}\frac{dy_5}{dx} = \frac{-ab}{(a+b+1)(1-x)-(a+b-c+1)} + \frac{(a+1)(b+1)(1-x)x}{(a+b+3)(1-x)-(a+b-c+2)} + \ldots\ldots$$

$$\ldots + \frac{(a+n)(b+n)(1-x)x}{(a+b+2n+1)(1-x)-(a+b-c+n+1)}$$

$$= \frac{-ab}{c-(a+b+1)x} - \frac{(a+1)(b+1)x(x-1)}{(c+1)-(a+b+3)x} - \ldots$$

$$\ldots - \frac{(a+n)(b+n)x(x-1)}{(c+n)-(a+b+2n+1)x} - \ldots$$

where $x=z^2$,

or

$$\frac{-ab}{c-(a+b+1)z^2} - \frac{(a+1)(b+1)z^2(z^2-1)}{(c+1)-(a+b+3)z^2} - \ldots - \frac{(a+n)(b+n)z^2(z^2-1)}{(c+n)-(a+b+2n+1)z^2} - \ldots\ldots$$

$$= -\frac{1}{y_5}\cdot\frac{dy_5}{dz}\cdot\frac{1}{2z}.$$

Thus the required relation becomes

$$\frac{-ab}{-(a+b+1)z^2 - \dfrac{(a+1)(b+1)z^2(z^2-1)}{\frac{3}{2}-(a+b+3)z^2} - \ldots - \dfrac{(a+n)(b+n)z^2(z^2-1)}{\frac{2n+1}{2}-(a+b+2n+1)z^2} - \ldots\ldots\ldots\ldots}$$

$$= -\frac{1}{2z}\left\{\frac{-4ab}{(2a+2b+1)z} - \frac{(2a+1)(2b+1)(z^2-1)}{(2a+2b+3)z} - \ldots - \frac{(2a+n)(2b+n)(z^2-1)}{(2a+2b+2n+1)z} - \ldots\right\}, \quad (17)$$

which is convergent in the domain of the singularity $z^2=1$.

§ 7. *Another Transformation of the P-function.*

We have seen (§ 5) that the P-function corresponding to the equation

$$(az^2+bz+c)\frac{d^2y}{dz} + (ez+f)\frac{dy}{dz} + gy = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

is

$$\mathrm{P}\left\{\begin{matrix} \lambda & \infty & \mu & \\ 0 & \beta & 0 & z \\ 1+\dfrac{e\lambda+f}{a(\mu-\lambda)} & \beta' & 1+\dfrac{e\mu+f}{a(\lambda-\mu)} & \end{matrix}\right\}.$$

The transformation of the dependent variable from y to u by the substitution

$$y = \left(\frac{z-\lambda}{z-\mu}\right)^{\delta} u$$

gives us the relation

$$\left(\frac{z-\lambda}{z-\mu}\right)^{\delta} \mathrm{P}\left\{\begin{matrix} \lambda & \infty & \mu & \\ 0 & \beta & 0 & z \\ 1+\dfrac{e\lambda+f}{a(\mu-\lambda)} & \beta' & 1+\dfrac{e\mu+f}{a(\lambda-\mu)} & \end{matrix}\right\}$$

$$= \mathrm{P}\left\{\begin{matrix} \lambda & \infty & \mu & \\ \delta & \beta & -\delta & z \\ 1+\dfrac{e\lambda+f}{a(\mu-\lambda)}+\delta & \beta' & 1+\dfrac{e\mu+f}{a(\lambda-\mu)}-\delta & \end{matrix}\right\} \quad \ldots\ldots(18)$$

This second P-function is of the same form as the first if

$$1+\frac{e\lambda+f}{a(\mu-\lambda)}+\delta = 0$$

and

$$1+\frac{e\mu+f}{a(\lambda-\mu)}-\delta = 0,$$

and these conditions are satisfied if

$$e = 2a,$$
$$f = -a(\lambda+\mu) + a\delta(\lambda-\mu).$$

When e and f have these particular values we are able to deduce a continued fraction-relation equivalent to the relation between the two P-functions.

The equation (8) becomes, on substituting for e and f their values as given above,

$$(az^2+bz+c)\frac{d^2y}{dz^2} + \{2az - a(\lambda+\mu) + a\delta(\lambda-\mu)\}\frac{dy}{dz} + gy = 0, \ldots\ldots\ldots(19)$$

from which we obtain the relation

$$\frac{1}{y}\frac{dy}{dz} = \cfrac{-g}{2az + a\delta(\lambda-\mu) - a(\lambda+\mu) - \cfrac{(2a+g)a(z-\lambda)(c-\mu)}{4az + a\delta(\lambda-\mu) - 2a(\lambda+\mu) - \ldots\ldots - \cfrac{\{n(n+1)a+g\}a(z-\lambda)(z-\mu)}{2(n+1)az + a\delta(\lambda-\mu) - (n+1)a(\lambda+\mu) - \ldots}}}$$

The transformed P-function has the form

$$\mathrm{P}\left\{\begin{array}{cccc} \lambda & \infty & \mu & \\ \delta & \beta & -\delta & z \\ 0 & \beta' & 0 & \end{array}\right\}$$

or

$$\mathrm{P}\left\{\begin{array}{ccccc} \lambda & & \infty & & \mu & \\ 0 & & \beta' & & 0 & z \\ 1+\dfrac{e\mu+f}{a(\lambda-\mu)} & & \beta & & 1+\dfrac{e\lambda+f}{a(\mu-\lambda)} & \end{array}\right\}$$

the equation corresponding to which is

$$(az^2+bz+c)\frac{d^2u}{dz^2} + \{2az - a(\lambda+\mu) - a\delta(\lambda-\mu)\}\frac{du}{dz} + gu = 0. \ldots\ldots (20)$$

whence we derive the relation

$$\frac{1}{u}\frac{du}{dz} = \cfrac{-g}{2az - a\delta(\lambda-\mu) - a(\lambda+\mu) - \cfrac{(2a+g)a(z-\lambda)(z-\mu)}{4az - a\delta(\lambda-\mu) - 2a(\lambda+\mu) - \ldots\ldots - \cfrac{\{n(n+1)a+g\}a(z-\lambda)(z-\mu)}{\ldots - 2(n+1)az - a\delta(\lambda-\mu) - (n+1)a(\lambda+\mu) - \ldots\ldots}}}$$

Now $y = \left(\dfrac{z-\mu}{z-\lambda}\right)^{\delta} u.$

Hence

$$\log y = \delta\log(z-\mu) - \delta\log(z-\lambda) + \log u,$$

and

$$\frac{1}{y}\frac{dy}{d} = \frac{\delta}{z-\mu} - \frac{\delta}{z-\lambda} + \frac{1}{u}\frac{du}{dz}$$

$$= \frac{\delta(\mu-\lambda)}{(z-\lambda)(z-\mu)} + \frac{1}{u}\frac{du}{dz}.$$

Thus we obtain the following relation between two continued fractions

$$\cfrac{-g}{2az-a\delta(\lambda-\mu)-a(\lambda+\mu)} - \cfrac{(2a+g)a(z-\lambda)(z-\mu)}{4az-a\delta(\lambda-\mu)-2a(\lambda+\mu)} - \ldots$$

$$\cfrac{\{n(n+1)a+g\}a(z-\lambda)(z-\mu)}{-2(n+1)az-a\delta(\lambda-\mu)-(n+1)a(\lambda+\mu)} - \ldots$$

$$= \frac{\delta(\lambda-\mu)}{(z-\lambda)(z-\mu)} + \cfrac{-g}{2az+a\delta(\lambda-\mu)-a(\lambda+\mu)} - \cfrac{(2a+g)a(z-\lambda)(z-\mu)}{4az+a\delta(\lambda-\mu)-2a(\lambda+\mu)} -$$

$$\ldots - \cfrac{\{n(n+1)a+g\}a(z-\lambda)(z-\mu)}{2(n+1)az+a\delta(\lambda-\mu)-(n+1)a(\lambda+\mu)} - \ldots \quad \ldots(21)$$

For the particular case of the ordinary hypergeometric equation

$$z(z-1)\frac{d^2y}{dz^2} + \{-z+(\alpha+\beta+1)z\}\frac{dy}{dz} + \alpha\beta y = 0$$

we have

$$a=1,\ \lambda=0,\ \mu=1,$$

$$-\gamma = f = -1-\delta,$$

$$\alpha+\beta+1 = e = 2,$$

and

$$\alpha\beta = g.$$

The relation between the continued fractions now becomes

$$\cfrac{-g}{2z+\delta-1} - \cfrac{(g+2)z(z-1)}{4z+\delta-2} - \cfrac{(g+6)z(z-1)}{6z+\delta-3} - \ldots - \cfrac{\{n(n+1)+g\}z(z-1)}{2(n+1)z+\delta-(n+1)} - \ldots$$

$$\cfrac{-\delta}{z(z-1)} + \cfrac{-g}{2z-\delta-1} - \cfrac{(g+2)z(z-1)}{4z-\delta-2} - \cfrac{(g+6)z(z-1)}{6z-\delta-3} - \ldots - \cfrac{\{n(n+1)+g\}z(z-1)}{2(n+1)z-\delta-(n+1)} - \ldots$$

$$(g \neq 0) \qquad \ldots(22)$$

The expansions of the two P-functions involved are the same in the two cases, as they correspond to the expansions at the singularity λ (or μ) with exponent zero, and thus the above equation (21) is valid, except at the points $z=\lambda$ and $z=\mu$.

§ 8. *The Hypergeometric Function.*

As is shown in text books on differential equations, *e.g.* Forsyth, § 117, the hypergeometric equation admits of 24 particular solutions: we have seen that there are six distinct solutions at various points in the z-plane, and consequently these 24 solutions may be collected into six groups of 4, the four solutions in each group being fundamentally the same.

Thus, for example, adopting the usual notation, we have

$$y_1 = y_2 = y_{17} = y_{18},$$

and these may be transformed into the corresponding members of the other groups.

Now $y_1 \equiv \mathrm{F}(a, b, c, z)$,

whence

$$\frac{1}{y_1}\frac{dy_1}{dz} = \frac{-ab}{(a+b+1)z-c} \frac{(a+1)(b+1)z(z-1)}{-(a+b+3)z-(c+1)} - \ldots - \frac{(a+n)(b+n)z(z-1)}{-(a+b+2n+1)z-(c+2)} - \ldots (2$$

also $y_2 \equiv (1-z)^{c-a-b}\mathrm{F}(c-a, c-b, c, z)$,

whence

$$\frac{1}{y_2}\frac{dy_2}{dz} = -\frac{c-a-b}{1-z} + \frac{-(c-a)(c-b)}{(2c-a-b+1)z-c} - \frac{(c-a+1)(c-b+1)z(z-1)}{(2c-a-b+3)z-(c+1)} - \ldots\ldots\ldots\ldots$$
$$\ldots - \frac{(c-a+n)(c-b+n)z(z-1)}{(2c-a-b+2n+1)z-(c+n)} - \ldots\ldots\ldots(2$$

Again, $y_{17} \equiv (1-z)^{-a}\mathrm{F}\left(a, c-b, c, \frac{z}{z-1}\right)$,

whence

$$\frac{1}{y_{17}}\frac{dy_{17}}{dz} = \frac{a}{1-z} - \frac{1}{z-1}\left\{\frac{-a(c-b)}{(a-b+1)z+c} - \frac{(a+1)(c-b+1)z}{(a-b+2)z+(c+1)} - \ldots\ldots\ldots\right.$$
$$\left. \ldots - \frac{(a+n)(c-z+n)z}{(a-b+n+1)z+(c+n)} - \ldots\right\} (2$$

similarly from $y_{18} \equiv (1-z)^{-b}\mathrm{F}\left(b, c-a, c, \frac{z}{z-1}\right)$, we obtain

$$\frac{1}{y_{18}}\frac{dy_{18}}{dz} = \frac{b}{1-z} - \frac{1}{z-1}\left\{\frac{-b(c-a)}{(b-a+1)z+c} - \frac{(b+1)(c-a+1)z}{(b-a+2)z+(c+1)} - \ldots\ldots\right.$$
$$\left. \ldots - \frac{(b+n)(c-a+n)z}{(b-a+n+1)z+(c+n)} - \ldots\right\} \ldots (2$$

Now these four solutions y_1, y_2, y_{17}, y_{18} are identical, and consequently the four continued fractions (23)—(26) derived from them by the same process in all cases are identical.

The hypergeometric series is, however, only convergent in the region $|z|<1$, so that at the point $z=1$ and outside the unit circle enclosing the origin the continued fraction relations given above do not hold. This limitation also applies in the cases of the relations between continued fractions derived from the hypergeometric equation in a later section of this paper.

§ 9. *Other relations deduced from hypergeometric functions.*

In addition to the identities between hypergeometric functions utilised in the preceding section, several important relations are known connecting different hypergeometric series, which would be expected to give rise to relations involving continued fractions. Thus from an equation given by Gauss, viz.

$$(1+z)^{2a}\mathrm{F}(2a,\ 2a+1-c,\ c,\ z)=\mathrm{F}\left(a,\ a+\tfrac{1}{2},\ c,\ \frac{4z}{(1+z)^2}\right),$$

is obtained the relation

$$\frac{a}{2(1-z)}-\frac{(1+z)}{4(1-z)}\left\{\frac{2a(2a+1-c)}{(4a+2-c)z-c}-\frac{(2a+1)(2a+2-c)z(z-1)}{(4a+4-c)z-(c+1)}-\ldots\right.$$
$$\left.\ldots-\frac{(2a+n)(2a+n-c)z(z-1)}{(4a+2n+2-c)z-(c+n)}-\ldots\right\}$$
$$=\frac{-a(a+\frac{1}{2})}{(2a+\frac{3}{2})4z-c(1+z)^2}+\frac{(a+1)(a+\frac{3}{2})4z(1-z)^2}{(2a+\frac{7}{2})4z-(c+1)(1+z)^2}+\ldots$$
$$\ldots+\frac{(a+n)\left(a+\dfrac{2n+1}{2}\right)4z(1-z)^2}{(2a+2n+\frac{3}{2})4z-(c+n)(1+z)^2}+\ldots\ldots(27)$$

Another equation given by Gauss, viz.

$$(1+z)^{2a}\mathrm{F}(a,\ a+\tfrac{1}{2}-b,\ b+\tfrac{1}{2},\ z^2)=\mathrm{F}\left(a,\ b,\ 2b,\ \frac{4z}{(1+z)^2}\right),$$

gives rise to the following continued fraction relation

$$\frac{a}{2(1-z)}+\frac{z(1+z)}{2(1-z)}\left\{\frac{-a(a-b-\frac{1}{2})}{(2a-b+\frac{3}{2})z^2-(b+\frac{1}{2})}-\frac{(a+1)(a-b+\frac{3}{2})z^2(z^2-1)}{(2a-b+\frac{7}{2})z^2-(b+\frac{3}{2})}-\ldots\ldots\right.$$

$$\left.\ldots-\frac{(a+n)\left(a-b+\frac{2n+1}{2}\right)z^2(z^2-1)}{(2a-b+2n+\frac{3}{2})z^2-\left(b+\frac{2n+1}{2}\right)}-\ldots\right\}$$

$$=\frac{-ab}{(a+b+1)4z-2b(1+z)^2}-\frac{(a+1)(b+1)4z(1-z)^2}{(a+b+3)4z-(2b+1)(1+z)^2}-\ldots\ldots\ldots\ldots$$

$$\ldots-\frac{(a+n)(b+n)4z(1-z)^2}{(a+b+2n+1)4z-(2b+n)(1+z)^2}-\ldots(2$$

Another relation between two continued fractions arises out of an equation give by Kummer, viz.

$$\mathrm{F}(a,\ b,\ a+b+\tfrac{1}{2},\ \sin^2\theta)=\mathrm{F}\left(2a,\ 2b,\ a+b+\tfrac{1}{2},\ \sin^2\frac{\theta}{2}\right),$$

from which we derive the following:—

$$\frac{2ab}{-(2a+2b+1)\cos\theta}+\frac{(2a+1)(2b+1)\sin^2\theta}{-(2a+2b+3)\cos\theta}+\ldots+\frac{(2a+n)(2b+n)\sin^2\theta}{-(2a+2b+2n+1)\cos\theta}+\ldots\ldots$$

$$=\cos\theta\left\{\frac{ab}{\frac{1}{2}-(a+b+1)\cos^2\theta}+\frac{(a+1)(b+1)\sin^2\theta\cos^2\theta}{\frac{3}{2}-(a+b+3)\cos^2\theta}+\ldots+\frac{(a+n)(b+n)\sin^2\theta\cos^2\theta}{\frac{2n+1}{2}-(a+b+2n+1)\cos^2\theta}+\right.$$

(

or if $\cos\theta=z$.

$$\frac{2ab}{-(2a+2b+1)z}+\frac{(2a+1)(2b+1)(1-z^2)}{-(2a+2b+3)z}+\ldots+\frac{(2a+n)(2b+n)(1-z^2)}{-(2a+2b+2n+1)z}+\ldots\ldots$$

$$=z\left\{\frac{ab}{\frac{1}{2}-(a+b+1)z^2}+\frac{(a+1)(b+1)z^2(1-z^2)}{\frac{3}{2}-(a+b+3)z^2}+\ldots+\frac{(a+n)(b+n)z^2(1-z^2)}{\frac{2n+1}{2}-(a+b+2n+1)z^2}+\ldots\right\}(3$$

when $|z|<1$.

Kummer also gives the following relation

$$\mathrm{F}\left(\frac{a}{2},\ \frac{a+1}{6},\ \frac{2a+2}{3},\ \sin^2 2\theta\right)=\cos^{-2a}\theta\,\mathrm{F}\left(\frac{a}{2},\ \frac{a+1}{6},\ \frac{2a+2}{3},\ -\frac{4\sin^2\theta}{\cos^4\theta}\right).$$

If $y = \mathrm{F}\left(\frac{a}{2}, \frac{a+1}{6}, \frac{2a+2}{3}, \sin^2 2\theta\right)$, the continued fraction corresponding to it is

$$\frac{1}{y}\frac{dy}{d(\sin^2 2\theta)} = \cfrac{\frac{-a(a+1)}{12}}{\left(\frac{2a}{3}+\frac{7}{6}\right)\sin^2 2\theta - \frac{2a+2}{3} + \cfrac{\frac{(a+2)(a+7)}{12}\sin^2 2\theta \cos^2 2\theta}{\left(\frac{2a}{3}+\frac{19}{6}\right)\sin^2 2\theta - \frac{2a+5}{3} + \dots}}$$

$$\dots + \cfrac{\frac{(a+2n)(a+6n+1)}{12}\sin^2 2\theta \cos^2 2\theta}{\left(\frac{2a}{3}+\frac{12n+7}{6}\right)\sin^2 2\theta - \frac{2a+3n+2}{3} + \dots}$$

$$= -\frac{1}{y}\frac{dy}{d(\cos^2\theta)} \quad \frac{1}{4(\cos^2\theta - \sin^2\theta)}.$$

Also, if $v = \mathrm{F}\left(\frac{a}{2}, \frac{a+1}{6}, \frac{2a+2}{3}, -\frac{4\sin^2\theta}{\cos^4\theta}\right)$ then the corresponding continued fraction is

$$\frac{dv}{d\left(\frac{-4\sin^2\theta}{\cos^4\theta}\right)} = \cfrac{\frac{-a(a+1)}{12}}{-\left(\frac{2a}{3}+\frac{7}{6}\right)\frac{4\sin^2\theta}{\cos^4\theta} - \frac{2a+2}{3} - \cfrac{\frac{(a+2)(a+7)}{12}\ \frac{4\sin^2\theta(\cos^2\theta-2)^2}{\cos^8\theta}}{-\left(\frac{2a}{3}+\frac{19}{6}\right)\frac{4\sin^2\theta}{\cos^4\theta} - \frac{2a+5}{3} - \dots\dots}}$$

$$\dots - \cfrac{\frac{(a+n)(a+6n+1)}{12} \cdot \frac{4\sin^2\theta(\cos^2\theta-2)^2}{\cos^8\theta}}{-\left(\frac{2a}{3}+\frac{12n+7}{6}\right)\frac{4\sin^2\theta}{\cos^4\theta} - \frac{2a+3n+2}{3} - \dots}$$

$$= -\frac{1}{4v}\frac{dv}{d(\cos^2\theta)}\frac{\cos^6\theta}{\cos^2\theta - 2}.$$

Now $$\frac{1}{y}\frac{dy}{d(\cos^2\theta)} = \frac{-a}{\cos^2\theta} + \frac{1}{v}\frac{dv}{d(\cos^2\theta)}$$

from which we may deduce another continued fraction relation, viz.

$$-4\cos 2\theta\left\{\frac{\frac{a(a+1)}{12}}{\left(\frac{2a}{3}+\frac{7}{6}\right)\sin^2 2\theta-\frac{2a+2}{3}}+\frac{\frac{(a+2)(a+7)}{12}\sin^2 2\theta\cos^2 2\theta}{\left(\frac{2a}{3}+\frac{19}{6}\right)\sin^2 2\theta-\frac{2a+5}{3}}+\ldots\ldots\ldots\ldots\right.$$

$$\left.+\frac{(a+2n)(a+6n+1)\sin^2 2\theta\cos^2 2\theta}{\left(\frac{2a}{3}+\frac{12n+7}{6}\right)\sin^2 2\theta-\frac{2a+3n+2}{3}}+\ldots\ldots\ldots\right\}$$

$$=\frac{a}{\cos^2\theta}+\frac{\cos^2\theta-2}{\cos^2\theta}\left\{\frac{\frac{-a(a+1)}{3}}{-\left(\frac{2a}{3}+\frac{7}{6}\right)4\sin^2\theta-\frac{2a+2}{3}\cos^4\theta-}\right.\frac{\frac{(a+2)(a+7)}{3}}{-\left(\frac{2a}{3}+\frac{19}{6}\right)4\sin^2\theta-\left(\frac{2a+3}{3}\right)\cos^4\theta-}$$

$$\left.-\ldots-\frac{\frac{(a+n)(a+6n+1)}{3}}{-\left(\frac{2a}{3}+\frac{12n+7}{6}\right)4\sin^2\theta-\left(\frac{2a+3n+2}{3}\right)\cos^4\theta-\ldots}\right\}. \quad (31$$

All the relations between continued fractions which are derived from equations between hypergeometric functions must be identically true, since an equation between the hypergeometric functions implies that they are the *same* solution of the *same* differential equation.

In conclusion, I must thank Professor Whittaker, at whose instigation this present research was begun, for encouragement and advice given to me during its progress.

The six trisectors of each of the angles of a triangle.

By F. Glanville Taylor and W. L. Marr.

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1. The following is an account of a theorem whose origin has been traced to Prof. Morley of Johns Hopkins University.

In the course of certain vector analysis, some 14 years ago, Prof. Morley found that if a variable cardioide touch the sides of a triangle the locus of its centre, that is, the centre of the circle on which the equal circle rolls, is a set of 9 lines which are three by three parallel, the directions being those of the sides of an equilateral triangle. The meets of these lines correspond to double tangents; they are also the meets of certain pairs of trisectors of the angles, internal and external, of the first triangle. This result was never published, and it was only the particular case of the internal trisectors that reached the present writers, the existence of the enveloping cardioides and the set of 9 lines being quite unknown to them.

It was only after the present paper had been read that the above information, thanks to various correspondents, was obtained. Hence the theorem has been approached from quite a different point of view, as also the extension of the particular case, which was erroneously thought to be unknown. Certain other properties which follow are, however, probably new.

The particular case will be given first, one proof being due to Mr W. E. Philip, H.M.I.S., from whom the theorem was originally obtained.

2. *Morley's Theorem. If the angles of any triangle ABC be trisected, the triangle DEF, formed by the meets of pairs of trisectors, each pair being adjacent to the same side of ABC, is equilateral.*

First geometrical proof (by W. E. Philip).

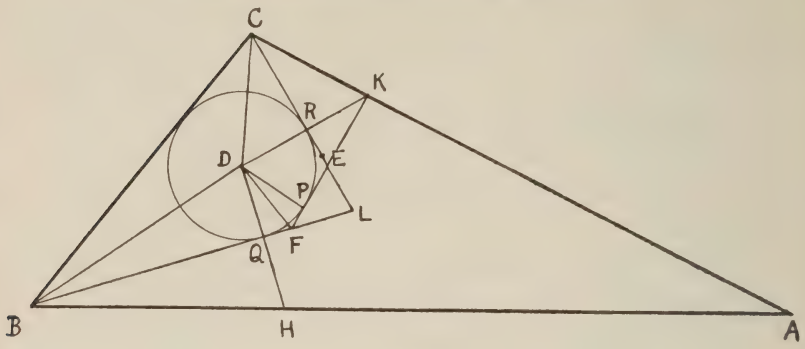

Fig. 1.

Let α, β, γ stand for $\frac{A}{3}, \frac{B}{3}, \frac{C}{3}$, respectively, so that $\alpha+\beta+\gamma=\frac{\pi}{3}$. The pairs of trisectors of B and C meet in D and L. Draw the incircle of LBC, centre D. Let H, K be the images of D in BL and CL. Draw the tangent KP and produce to meet BL in F ; Q is the point of contact of BL.

Since $DP=\frac{1}{2}DK$, $\therefore$ $\angle PKD=\frac{\pi}{6}$

$$\angle PDK=\frac{\pi}{3}$$

$$QDR=\pi-L=2\beta+2\gamma=\frac{2\pi}{3}-2\alpha$$

$$\therefore \quad FHQ=FDQ=\tfrac{1}{2}QDP=\tfrac{1}{2}\left(QDR-\frac{\pi}{3}\right)$$

$$=\frac{\pi}{6}-\alpha.$$

Also $$DHK=DKH=\tfrac{1}{2}L=\frac{\pi}{6}+\alpha$$

whence $$FHK=2\alpha, \quad FKH=\alpha.$$

Hence $HFK=\pi-3\alpha=\pi-A$, and AHFK is cyclic.

$\therefore$ $HAF=HKF=\alpha$, and AF is a trisector of A. Similarly for E, the meet of CL and the tangent from H.

From the symmetry of the figure DHK with respect to the circle it follows that $DF = DE$ and $\angle FDE$ is easily seen to be $\frac{\pi}{3}$. $\therefore$ $\triangle DEF$ is equilateral.

3. *Second geometrical proof.*

(i) Let AD, BE, CF meet BC, CA, AB in X, Y, Z respectively, also let X_1, Y_1, Z_1; X_2, Y_2, Z_2; X_3, Y_3, Z_3 be the feet of perpendiculars from D, E, F respectively to the sides of ABC.

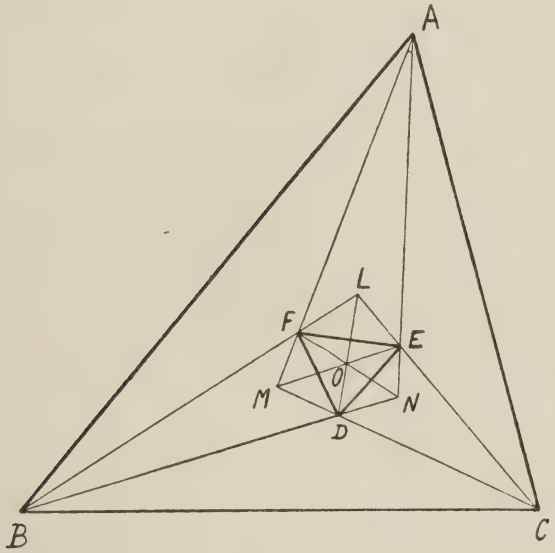

Fig. 2.

Then $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{\triangle ADB}{\triangle ADC} \cdot \frac{BEC}{BEA} \cdot \frac{CFA}{CFB}$

$= \frac{DZ_1}{DY_1} \cdot \frac{EX_2}{EZ_2} \cdot \frac{FY_3}{FX_3} = 1$, since by similar triangles

$\frac{EZ_2}{EY_2} = \frac{FY_3}{FZ_3}$; $\frac{FX_3}{FZ_3} = \frac{DZ_1}{DX_1}$; $\frac{DY_1}{DX_1} = \frac{EX_2}{EY_2}$. [Or, by Trilinears.]

Hence AD, BE, CF are concurrent.

(ii) It is evident that AFBDCE is a Brianchon hexagon ; and taking the sides in the orders AMCLBN, LFMDNE respectively, it follows that AL, BM, CN are concurrent, as also DL, EM, FN. Let the latter point be O.

(iii) In the quadrilateral OMDN, since MO and NO bisect the angles at M and N, we easily obtain

$$\angle \mathrm{OMD} = \frac{\pi}{6} + \frac{\mathrm{B}}{3};\ \mathrm{OND} = \frac{\pi}{6} + \frac{\mathrm{C}}{3};\ \text{and}\ \mathrm{MDN} = \frac{2\pi}{3} + \frac{\mathrm{A}}{3};$$

whence $\mathrm{MON} = \frac{2\pi}{3},\ \mathrm{NOE} = \mathrm{FOM} = \frac{\pi}{3}.$

Hence the acute angles at O are each equal to $\frac{\pi}{3}$, the $\triangle^{\text{les}}$ FOL, EOL are congruent, and LF = LE.

Finally, FDL, EDL are congruent and DF = DE ; whence it follows that DEF is equilateral.

The concurrency of DL, EM, FN may be proved by trilinears thus :—

Since D is the point $\left(1,\ 2\cos\frac{\mathrm{C}}{3},\ 2\cos\frac{\mathrm{B}}{3}\right)$, its isogonal conjugate L is the point $\left(1, 1/2\cos\frac{\mathrm{C}}{3}, 1/2\cos\frac{\mathrm{B}}{3}\right)$ or $\left(2\cos\frac{\mathrm{B}}{3}\cos\frac{\mathrm{C}}{3},\ \cos\frac{\mathrm{B}}{3}, \cos\frac{\mathrm{C}}{3}\right)$.

Multiplying the coordinates of D by $\cos\frac{\mathrm{A}}{3}$ and adding those of L, we get the point (on DL)

$$(2\cos\beta\cos\gamma + \cos\alpha,\ 2\cos\gamma\cos\alpha + \cos\beta,\ 2\cos\alpha\cos\beta + \cos\gamma),$$

where $\alpha = \frac{\mathrm{A}}{3}$, etc.

By symmetry it is seen that this point lies on EM and FN ; it is therefore the point O.

4. *Proof by Trigonometry.*

$$\frac{\sin\mathrm{AEF}}{\sin\mathrm{AFE}} = \frac{\mathrm{AF}}{\mathrm{AE}} = \frac{c\sin\beta}{\sin\left(\frac{\pi}{3} - \gamma\right)} \cdot \frac{\sin\left(\frac{\pi}{3} - \beta\right)}{b\sin\gamma} = \frac{\sin\left(\frac{\pi}{3} + \gamma\right)}{\sin\left(\frac{\pi}{3} + \beta\right)}.$$

But $\frac{\pi}{3} + \beta + \frac{\pi}{3} + \gamma = \pi - \alpha = \mathrm{AFE} + \mathrm{AEF},$

$\therefore\ \mathrm{AFE} = \frac{\pi}{3} + \beta\ ;\ \ \mathrm{AEF} = \frac{\pi}{3} + \gamma.$

Similarly $\mathrm{BFD} = \frac{\pi}{3} + \alpha$; $\mathrm{BDF} = \frac{\pi}{3} + \gamma$; etc.

Hence $\mathrm{DFE} = \frac{\pi}{3}$, and similarly for the other angles of DEF.

Also $$\mathrm{EF} = \frac{\mathrm{AF}\sin\alpha}{\sin\left(\frac{\pi}{3}+\gamma\right)} = \frac{2\mathrm{R}\sin\mathrm{C}\sin\beta}{\sin\left(\frac{\pi}{3}-\gamma\right)} \cdot \frac{\sin\alpha}{\sin\left(\frac{\pi}{3}+\gamma\right)} = 8\mathrm{R}\sin\alpha\sin\beta\sin\gamma.$$

The angle $(\mathrm{BC}, \mathrm{FE}) = \mathrm{BCA} + \mathrm{CAE} - \mathrm{AEF}$, negatively

$$= \mathrm{C} + \alpha - \frac{\pi}{3} - \gamma$$

$$= 2\gamma + \alpha - (\alpha + \beta + \gamma) = \gamma - \beta.$$

Hence the directions of FE, DF, ED make angles $\beta - \gamma$, $\gamma - \alpha$, $\alpha - \beta$ with the sides BC, CA, AB respectively.

5. *Extension of theorem.*

Suppose now we take a more general view of the triangle ABC. Let the angles be $\mathrm{A} + 2p\pi$, $\mathrm{B} + 2q\pi$, $\mathrm{C} + 2r\pi$. The angles BAF, CAE will be measured in the counterclockwise and clockwise directions respectively, and similarly for the angles at B and C. Thus a rotation of the sides inwards such as BC to BD, CB to CD will be in the positive direction.

Let $\frac{\mathrm{A}}{3} + \frac{2p\pi}{3}$ be represented by α_p, etc. Then for each of the six original lines BD, CD, etc., we shall have three, giving 6 at each vertex. There will be 9 points D, and 27 in all. The general triangle will be denoted by $\mathrm{D}_{qr}\mathrm{E}_{rp}\mathrm{F}_{pq}$ or $[pqr]$, where p, q, r take any of the values 0, 1, 2. We shall show that the 27 points D, E, F lie six by six on 9 straight lines, which are 3 by 3 parallel, and inclined at 60°. We thus get 27 equilateral triangles, of which 18 correspond to $\mathrm{D}_{00}\mathrm{E}_{00}\mathrm{F}_{00}$, the remaining 9 involving D's, E's or F's only. There are also 9 non-equilateral triangles DEF which are closely allied to the 18 referred to above.

D_{qr}, then, denotes the point whose biangular coordinates referred to B and C are (β_q, γ_r). It will be helpful to remember that q refers to the *left* vertex B, and r to the *right* vertex C, as we face BC, etc. [See Fig. 3.]

The coordinates of D_{00} are $(1, 2\cos\gamma_0, 2\cos\beta_0)$ which we shall denote by $(1\gamma_0\beta_0)$. Hence we easily arrive at the following table

for D_{qr}, or $(1, 2\cos\gamma_r, \cos\beta_q)$, or $(1\gamma_r\beta_q)$, from which we can deduce E_{rp}, F_{pq}.

$D_{00}(1\gamma_0\beta_0)$	$E_{00}(\gamma_0 1\alpha_0)$	$F_{00}(\beta_0\alpha_0 1)$
$D_{01}(1\gamma_1\beta_0)$	$E_{01}(\gamma_0 1\alpha_1)$	$F_{01}(\beta_1\alpha_0 1)$
etc.	etc.	etc.

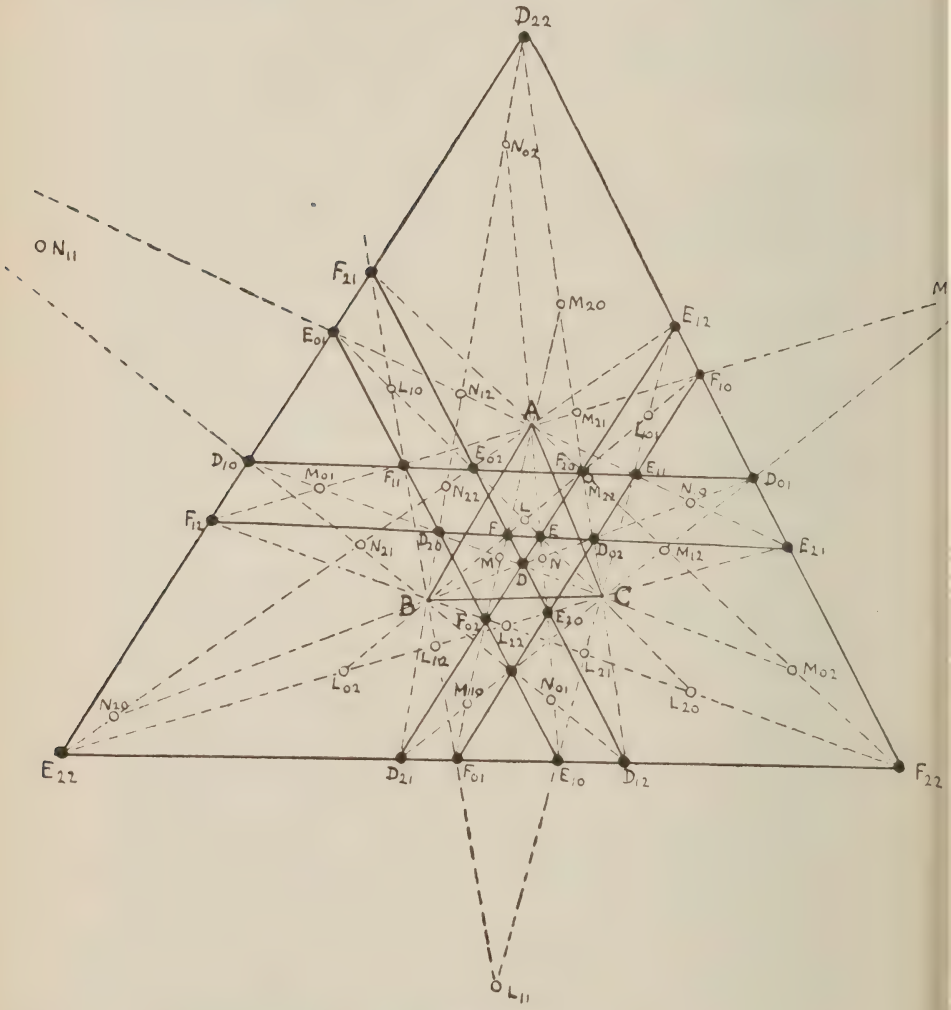

Fig. 3.

6. The equation to $E_{00}F_{00}$ is reducible to the form

$$\alpha\sin\alpha_2 + \beta\sin\beta_1 + \gamma\sin\gamma_1 = 0 \quad \ldots\ldots(1)$$

We shall show that besides E_{00} and F_{00} there are four other points D_{02}, F_{12}, D_{20}, E_{21} on this same line. We might, of course, merely substitute coordinates, but this is tedious and does not enable us to discover the points. We shall, instead, interchange the letters, increasing and diminishing suffixes so that the sum of the angles remains equal to $\alpha_2 + \beta_1 + \gamma_1$ or differs from it by 2π: also we shall make cyclical changes so as to bring the equation back to its original form. In the case of points on the line a corresponding cyclical change is made in the coordinates including the order as well as the letter.

Thus from $E_{00}(\gamma_0 1 \alpha_0)$ we obtain $F_{00}(\beta_0 \alpha_0 1)$ by interchanging β and γ and the order of the β and γ coordinates $(1\alpha_0)$, the equation (1) remaining unaltered by the change.

Next change α_p, β_q to α_{p-1}, β_{q+1} respectively, and make a negative cyclical change in α, β, γ. Thus (1) becomes itself again; and the point $(\gamma_0 1 \alpha_0)$ becomes first $(\gamma_0 1 \alpha_2)$ since $2 \equiv -1 (\text{mod. } 3)$, and afterwards, making a negative cyclical change in the order as well as in the letters, $(1\gamma_2\beta_0)$, which is D_{02}.

Interchanging β and γ we obtain $(1\gamma_0\beta_2)$ or D_{20}.

Similarly by changing α_p, γ_r to α_{p-1}, γ_{r+1} respectively in (1), and making a positive cyclical change in the letters and their order we obtain $(\beta_2\alpha_1 1)$ or F_{12}; and finally $(\gamma_2 1 \alpha_1)$ or E_{21}.

Hence there are six points on $E_{00}F_{00}$, and similarly for $F_{00}D_{00}$ and $D_{00}E_{00}$. The other sets of six are easily obtained by increasing all the suffixes by 1, and again by 1.

Thus $E_{11}F_{11}$ is $\alpha\sin\alpha_0 + \beta\sin\beta_2 + \gamma\sin\gamma_2 = 0 \quad \ldots\ldots(2)$

$E_{22}F_{22}$ is $\alpha\sin\alpha_1 + \beta\sin\beta_0 + \gamma\sin\gamma_0 = 0 \quad \ldots\ldots(3)$

In (2) the sum of the angles is unaltered, and in (3) it is diminished by 2π.

It can be shown by elementary geometry that these three lines are parallel. For example, the triangles AEC, ABF_{01} are similar, as also are AFB, ACE_{10}; hence

$$\frac{AE}{AC} = \frac{AB}{AF_{01}}, \quad \frac{AF}{AB} = \frac{AC}{AE_{10}}; \quad \therefore \quad \frac{AE}{AF} = \frac{AE_{10}}{AF_{01}}$$ and EF is $\parallel$ to $E_{10}F_{01}$, etc.

7. To each of the 9 D's correspond three triangles DEF, of which only two, however, are equilateral. Thus keep D_{00} fixed;

then since β_0 and γ_0 are fixed the lines BL, CL are fixed. Change α_0 to α_1 and α_2 in succession, *i.e.* rotate AE and AF (in their proper directions, see Art. 5) through $\frac{2\pi}{3}$ twice in succession and we shall obtain the triangles $DE_{01}F_{10}$ and $DE_{02}F_{20}$. Corresponding to DL, EM, FN we shall have DL, $E_{01}M_{01}$, $F_{10}N_{10}$ and DL, $E_{02}M_{02}$, $F_{20}N_{20}$. It can be shewn without difficulty that while DL is fixed, EM and FN are turned in opposite directions through $\frac{\pi}{3}$, so that DL, $E_{01}M_{01}$, $F_{10}N_{10}$ are parallel and the proof breaks down, though not again in the third triangle.

It will be seen that the triangle $[pqr]$ is equilateral provided that $p+q+r \neq 1 \pmod 3$, for increasing one of the numbers by unity corresponds geometrically to rotating a pair of the lines DL, EM, FN through 60° as above. The nine non-equilaterals are [100], [010], [001], [211], [121], [112], [022], [202], [220]. Thus [022] is the triangle $D_{22}E_{20}F_{02}$.

The points L, M, N.

8. The points L, M, N, which are the isogonal conjugates of D, E, F, are 27 in number. Noting that the locus of the isogonal conjugates of points on the line $p\alpha+q\beta+r\gamma=0$ is the conic $p\beta\gamma+q\gamma\alpha+r\alpha\beta=0$,—the line at infinity, in particular, leading to the circumcircle,—it follows that the 27 points L, M, N lie 6 by 6 on 9 conics through ABC; and that these meet three by three on the circumcircle. It may be shown that the three meeting points form an equilateral triangle whose sides are parallel to those of DEF. We give a short proof:—

Draw through A lines Ad, Ae, Af parallel to EF, FD, DE to meet the circumcircle. Then since the isogonal lines Ad', Ae', Af' make respectively the same angles with AI (I the incentre) as these, but in the opposite sense, it is evident that both *def* and *d'e'f'* are equilateral triangles.

Writing x for $\gamma_0 - \beta_0$ or $\frac{C-B}{3}$, we have, if $x = (d\text{A}, \text{BC})$,

$$\angle \text{BA}d' = \text{CAX} = \text{C} - x \quad \text{(Art. 4)}.$$
$$\text{BA}d = \text{B} + x.$$

$\therefore$ $d\text{A}d' = \text{C} - \text{B} - 2x = x = \text{AC}d'$, since C$d'$ is isogonal to Cy ($\parallel$ AX), $= \text{A}dd'$

or d' is the mid point of the arc Ad.

Hence $e'f'$ is parallel to Ad or EF.
Similarly for $f'd'$ and $d'e'$.

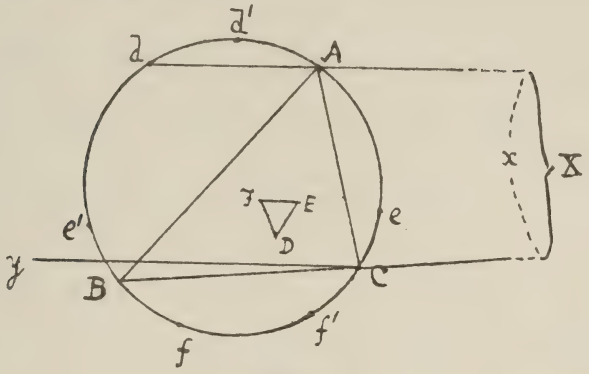

Fig. 4.

9. The lines AL_{qr}, BM_{rp}, CN_{pq} are concurrent; for, referring to the table of Art. 5, we find that L_{qr}, the isogonal conjugate of D_{qr} is $(1, 1/\gamma_r, 1/\beta_q)$ or $(2\cos\beta_q\cos\gamma_r, \cos\beta_q, \cos\gamma_r)$.

It is thus seen that the point of concurrence is $(\cos\alpha_p, \cos\beta_q, \cos\gamma_r)$. We shall denote this by (pqr).

Corresponding to the 27 triangles DEF are 27 points (pqr).

10. We shall now show (1) that these points (pqr) are the poles of 27 pairs of triangles DEF, comprising the 18 equilateral triangles and the 9 non-equilaterals referred to above; (2) that the former consist of 6 triads of equilaterals, each triad being two by two in perspective; the latter of 3 such triads; and (3) that the poles of each triad are collinear; (4) each line $D_{qr}\,D_{q+1,\,r+1}$, etc., passes through three of the points (pqr).

(1) Take a particular case, $D_{00}(1\,2\cos\gamma_0\,2\cos\beta_0)$, $D_{11}(1\,2\cos\gamma_1\,2\cos\beta_1)$. Multiply the coordinates of D_{00} by $\sin\alpha_0$, those of D_{11} by $-\sin\alpha_1$, and add.

Then $\sin\alpha_0 - \sin\alpha_1 = 2\cos\alpha_2\sin\frac{\pi}{3}$.

$2\cos\gamma_0\sin\alpha_0 - 2\cos\gamma_1\sin\alpha_1 = 2\cos\beta_0\sin\frac{\pi}{3}$ eventually,

(see the general case).

$2\cos\beta_0\sin\alpha_0 - 2\cos\beta_1\sin\alpha_1 = 2\cos\gamma_0\sin\frac{\pi}{3}$.

Hence $D_{00}D_{11}$ passes through (200).

(2) *Generally*, $D_{qr}\,(1, 2\cos\gamma_r, 2\cos\beta_q)$, $D_{q+1,\,r+1}(1, 2\cos\gamma_{r+1}, 2\cos\beta_{q+1})$.
Take as multiples $\sin\alpha_m$, $-\sin\alpha_n$, $(m \neq n)$.

Then (a) $\sin\alpha_m - \sin\alpha_n = 2\cos\left\{\alpha + (m+n)\frac{\pi}{3}\right\}\sin(m-n)\frac{\pi}{3}$,

(b) $2\cos\gamma_r\sin\alpha_m - 2\cos\gamma_{r+1}\sin\alpha_n = \sin\left\{\gamma + \alpha + (m+r)\frac{2\pi}{3}\right\}$

$$- \sin\left\{\gamma - \alpha + (r-m)\frac{2\pi}{3}\right\} - \sin\left\{\gamma + \alpha + (r+1+n)\frac{2\pi}{3}\right\}$$

$$+ \sin\left\{\gamma - \alpha + (r+1-n)\frac{2\pi}{3}\right\}.$$

Now let $(r+1-n)\frac{2\pi}{3} - (r-m)\frac{2\pi}{3} = 2k\pi$.

$\therefore \quad m = n + 3k - 1.$

$\therefore$ L.H.S. $= 2\cos\left\{\gamma + \alpha + (m+n+2r+1)\frac{\pi}{3}\right\}\sin(m-n-1)\frac{\pi}{3}$

$$= 2\cos\left\{\beta - (2n+2r+3k+1)\frac{\pi}{3}\right\}\sin(3k-2)\frac{\pi}{3}$$

$$= 2\cos\left\{\beta - (2n+2r+1)\frac{\pi}{3}\right\}\sin\left(-\frac{2\pi}{3}\right).$$

$$= 2\cos\left\{\beta - (n+r-1)\frac{2\pi}{3}\right\}\sin\frac{\pi}{3} = 2\cos\left\{\beta + (4-n-r)\frac{2\pi}{3}\right\}\sin\frac{\pi}{3}$$

$$= 2\cos\beta_{4-n-r} \cdot \sin\frac{\pi}{3}.$$

Similarly $2\cos\beta_q\sin\alpha_m - 2\cos\beta_{q+1}\sin\alpha_n = 2\cos\gamma_{4-n-q} \cdot \sin\frac{\pi}{3}$.

Also in (a) $\sin\alpha_m - \sin\alpha_n = 2\cos\left\{\alpha + (2n+3k-1)\frac{\pi}{3}\right\}\sin(3k-1)\frac{\pi}{3}$

$$= 2\cos\left\{\alpha + (2n-1)\frac{\pi}{3}\right\}\sin\left(-\frac{\pi}{3}\right) = 2\cos\left\{\alpha + (n+1)\frac{2\pi}{3}\right\}\sin\frac{\pi}{3}$$

$$= 2\cos\alpha_{n+1} \cdot \sin\frac{\pi}{3}.$$

Thus $D_{qr}\,D_{q+1,\,r+1}$ passes through $(n+1, 4-n-r, 4-n-q)$.*

---

* Or, if preferred, $D_{q-1,\,r-1}\,D_{qr}$ passes through the point $(-n, n-r, n-q)$.

Since n, q, r may have each 3 values, there are 27 points. But there are only 9 lines $D_{qr}\,D_{q+1,\,r+1}$, since n is not involved here.

We tabulate the results as follows:—

Collinear triads of points.

q	r	$D_{qr}D_{q+1,\,r+1}$	$n=0$ (1, 4 − r, 4 − q)	$n=1$ (2, 3 − r, 3 − q)	$n=2$ (0, 2 − r, 2 − q)
0	0	$D_{00}\,D_{11}$	(111)	(200)	(022)
0	1	$D_{01}\,D_{12}$	(101)	(220)	(012)
0	2	$D_{02}\,D_{10}$	(121)	(210)	(002)
1	0	$D_{10}\,D_{21}$	(110)	(202)	(021)
1	1	$D_{11}\,D_{22}$	(100)	(222)	(011)
1	2	$D_{12}\,D_{20}$	(120)	(212)	(001)
2	0	$D_{20}\,D_{01}$	(112)	(201)	(020)
2	1	$D_{21}\,D_{02}$	(102)	(221)	(010)
2	2	$D_{22}\,D_{00}$	(122)	(211)	(000)

Similar tables for EE and FF may be obtained merely by a cyclical change in the numbers (pqr), giving (qrp) and (rpq) respectively.

Thus	$E_{20}E_{01}$	gives	(211)	(120)	(002)
	$F_{20}F_{01}$	,,	(121)	(012)	(200)

11. Since in each of these cases we obtain the same 27 numbers again, but in different orders, it is clear that lines of the type DD, EE, FF are concurrent, as in the following table:—

*(022)	$D_{00}D_{11}$	$E_{01}E_{12}$	$F_{10}F_{21}$	(021)	$D_{10}D_{21}$	$E_{02}E_{10}$	$F_{21}F_{02}$	(020)	$D_{20}D_{01}$	$E_{00}E_{11}$	$F_{02}F_{10}$
(111)	$D_{00}D_{11}$	$E_{00}E_{11}$	$F_{00}F_{11}$	(110)	$D_{10}D_{21}$	$E_{01}E_{12}$	$F_{11}F_{22}$	*(112)	$D_{20}D_{01}$	$E_{02}E_{10}$	$F_{22}F_{00}$
(200)	$D_{00}D_{11}$	$E_{02}E_{10}$	$F_{20}F_{01}$	*(202)	$D_{10}D_{21}$	$E_{00}E_{11}$	$F_{01}F_{12}$	(201)	$D_{20}D_{01}$	$E_{01}E_{12}$	$F_{12}F_{20}$
(011)	$D_{11}D_{22}$	$E_{10}E_{21}$	$F_{01}F_{12}$	*(010)	$D_{21}D_{02}$	$E_{11}E_{22}$	$F_{12}F_{20}$	(012)	$D_{01}D_{12}$	$E_{12}E_{20}$	$F_{20}F_{01}$
*(100)	$D_{11}D_{22}$	$E_{12}E_{20}$	$F_{21}F_{02}$	(102)	$D_{21}D_{02}$	$E_{10}E_{21}$	$F_{02}F_{10}$	(101)	$D_{01}D_{12}$	$E_{11}E_{22}$	$F_{10}F_{21}$
(222)	$D_{11}D_{22}$	$E_{11}E_{22}$	$F_{11}F_{22}$	(221)	$D_{21}D_{02}$	$E_{12}E_{20}$	$F_{22}F_{00}$	*(220)	$D_{01}D_{12}$	$E_{10}E_{21}$	$F_{00}F_{11}$
(000)	$D_{22}D_{00}$	$E_{22}E_{00}$	$F_{22}F_{00}$	(002)	$D_{02}D_{10}$	$E_{20}E_{01}$	$F_{00}F_{11}$	*(001)	$D_{12}D_{20}$	$E_{21}E_{02}$	$F_{11}F_{22}$
(122)	$D_{22}D_{00}$	$E_{21}E_{02}$	$F_{12}F_{20}$	*(121)	$D_{02}D_{10}$	$E_{22}E_{00}$	$F_{20}F_{01}$	(120)	$D_{12}D_{20}$	$E_{20}E_{01}$	$F_{01}F_{12}$
*(211)	$D_{22}D_{00}$	$E_{20}E_{01}$	$F_{02}F_{10}$	(210)	$D_{02}D_{10}$	$E_{21}E_{02}$	$F_{10}F_{21}$	(212)	$D_{12}D_{20}$	$E_{22}E_{00}$	$F_{21}F_{02}$

Each line gives a pair of triangles in perspective, found by taking respectively the first or second letter of each pair DD, etc. Thus (011) is the pole of $D_{11}E_{10}F_{01}$ and $D_{22}E_{21}F_{12}$, or [011] and [122] using *square* brackets for triangles.

The rows marked * give the three triads of non-equilaterals, one triad in each column ; the remaining rows give 6 triads, two in each column, of equilaterals. The following analysis of the first column will explain an otherwise curious point :—

Triangle.				Pole.
[000]	D_{00}	E_{00}	F_{00}	(111)
[111]	D_{11}	E_{11}	F_{11}	(222)
[222]	D_{22}	E_{22}	F_{22}	(000)
[000]	D_{00}	E_{00}	F_{00}	

Poles asymmetrical (+)

Triangle.				Pole.
*[100]	D_{00}	E_{01}	F_{10}	(022)
*[211]	D_{11}	E_{12}	F_{21}	(100)
*[022]	D_{22}	E_{20}	F_{02}	(211)
*[100]	D_{00}	E_{01}	F_{10}	

Poles symmetrical (0)

Triangle.				Pole.
[200]	D_{00}	E_{02}	F_{20}	(200)
[011]	D_{11}	E_{10}	F_{01}	(011)
[122]	D_{22}	E_{21}	F_{12}	(122)
[200]	D_{00}	E_{02}	F_{20}	

Poles asymmetrical (−)

See Art. 7. Note that in the non-equilaterals (only) if (lmn) denote the pole of $[pqr]$, $[p+1, q+1, r+1]$, then $l+m+n \equiv p+q+r \equiv 1 \pmod 3$.

12. The fact that the three poles of any triad of triangles are collinear follows at once from a theorem of Steiner's ; for, taking the example given above in the first column, $E_{00}F_{00}$, $E_{11}F_{11}$, $E_{22}F_{22}$ meet at the same point on the line at infinity ; similarly for the lines FD, DE respectively. Hence the poles of the three pairs of triangles are collinear, as also follows, of course, from the fact that the determinant

$$\begin{vmatrix} \cos\alpha_p & \cos\beta_q & \cos\gamma_r \\ \cos\alpha_{p+1} & \cos\beta_{q+1} & \cos\gamma_{r+1} \\ \cos\alpha_{p+2} & \cos\beta_{q+2} & \cos\gamma_{r+2} \end{vmatrix} \equiv 0.$$

The Relation of Morley's Theorem to the Hessian Axis and Circumcentre.

By F. Glanville Taylor.

(*Read 8th May 1914. Received 23rd May 1914*).

1. In a previous paper (*q.v.*) by Mr W. L. Marr and the present writer, it was shown that, in accordance with Morley's Theorem, if the angles $A+2p\pi$, $B+2q\pi$, $C+2r\pi$ of the triangles ABC be trisected, the three groups of six lines at the vertices give rise to 27 triangles DEF, the biangular coordinates of D with respect to BC being $(B/3+2q\pi/3,\ C/3+2r\pi/3)$ or $(\beta_q,\ \gamma_r)$, and similarly for E and F with respect to CA and AB.

These 27 triangles, 18 of which are equilateral, were arranged in 9 triads, those of each triad being two by two in perspective, and the three poles being collinear. The coordinates of these poles were represented by

$$(pqr),\ (p+1,\ q+1,\ r+1),\ (p+2,\ q+2,\ r+2)\ldots\ldots\ldots\ldots(A)$$

where $(pqr)\equiv(\cos\alpha_p,\ \cos\beta_q,\ \cos\gamma_r)$.

Further, these 27 poles were shown to be points of concurrence of lines AL, BM, CN; where L, M, N are the isogonal conjugates of D, E, F, being in fact the meets of pairs of the second trisectors of the generalised angles $A+2p\pi$, etc.

The 9 lines DD each contain a triad of points (pqr); similarly for EE and FF.

We shall denote a progressive triad such as (A) by $\overline{pqr}$. In discussing the properties of these triads we depend entirely on Ptolemy's Theorem expressed in the forms

$$\Sigma\sin\lambda\sin(\mu-\nu)=0\ ;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\Sigma\cos\lambda\sin(\mu-\nu)=0.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

2. *The nine triads of poles.*

From (2) it follows that the points (A) lie on the line

$$\alpha\sin(\beta_q-\gamma_r)+\beta\sin(\gamma_r-\alpha_p)+\gamma\sin(\alpha_p-\beta_q)=0\ldots\ldots\ldots(3)$$

where $\beta_q-\gamma_r=\beta-\gamma+(q-r)2\pi/3\equiv(\beta-\gamma)_{q-r}\equiv L_i$, say.

(3) may therefore be written

$$\alpha \sin L_l + \beta \sin M_m + \gamma \sin N_n = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Since $l + m + n = 0$, there are only 9 different lines of this series formed by varying l, m, n, as is otherwise obvious.

3. *Three triads of concurrent lines.*

Again, a triad of lines $\overline{lmn}$ will by (1), pass through the point

$$[\sin(M_m - N_n), \sin(N_n - L_l), \sin(L_l - M_m)]$$

or

$$(\sin D_d, \sin E_e, \sin F_f) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where $D = M - N = \beta + \gamma - 2\alpha = \pi/3 - A = \pi - (A + 2\pi/3) \equiv \pi - A_1$,

and $d = m - n = q + r - 2p = p + q + r - 3p \equiv p + q + r \pmod 3$.

Hence $d \equiv e \equiv f \pmod 3$

and $D_d = (\pi - A_1)_d = \pi - A_{1-d}$.

The points (5) are therefore

$$(\sin A_{1-d}, \sin B_{1-d}, \sin C_{1-d}) \quad (d = 0, 1, 2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

These three points are

$(\sin A)$, the Lemoine or symmedian point,

$[\sin(A + 2\pi/3)]$, the negative Hessian point,

$[\sin(A + 4\pi/3)]$, the positive Hessian point.

It will be observed that when d, or $p + q + r$, is 1, the point is the Lemoine point. This corresponds to the group of 9 non-equilaterals DEF (see previous paper).

It is a curious fact, proved by Mr Marr, that the lines of the triads which meet at the Hessian points make angles $2\pi/3$ with each other in both cases.

4. *One triad of collinear points.*

Finally from (1) follows the well-known fact that these three points lie on the line

$$\alpha \sin(B - C) + \beta \sin(C - A) + \gamma \sin(A - B) = 0,$$

which is termed the Hessian axis.

5. *The lines DD, EE, FF.*

The line D_{qr}, $D_{q+1,\, r+1}$ was shown in the previous paper to contain the triad of points

$$(n + 1, 4 - n - r, 4 - n - q) \quad (n = 0, 1, 2).$$

There are 9 collinear triads, as may be seen by varying q and r, and as all the 27 points (pqr) are accounted for, we may, without confusion, express each triad in the form

$$(pqr),\ (p+1,\ q-1,\ r-1),\ (p+2,\ q-2,\ r-2).$$

Now let $\alpha = \alpha',\ \beta = -\beta',\ \gamma = -\gamma'$

$$p = p',\ q = -q',\ r = -r'.$$

Then $\cos\beta_q = \cos\beta'_{q'}$; $\cos\beta_{q-1} = \cos\beta'_{q'+1}$; $\cos\beta_{q-2} = \cos\beta'_{q'+2}$; and similarly for γ_r.

Hence the triad becomes the progressive triad $\overline{p'q'r'}$.

Proceeding as before, we obtain the nine lines DD, namely,

$$\Sigma\alpha\sin L'_{\nu} = 0$$

passing three by three through three points

$$(\sin D'_{d'},\ \sin E'_{d'},\ \sin F'_{d'}) \quad (d' = 0,\ 1,\ 2).$$

And these three points lie on the line

$$\Sigma\alpha\sin(E' - F') = 0$$

where $E' - F' = M' + N' - 2L' = L' + M' + N' - 3L' = -3L'$

$$= -3(\beta' - \gamma') = B - C$$

$$F' - D' = -3M' = -3(\gamma' - \alpha') = C + A = \pi - B$$

$$D' - E' = -3N' = -3(\alpha' - \beta') = -(A + B) = C - \pi.$$

The line is therefore

$$\alpha\sin(B - C) + \beta\sin B - \gamma\sin C = 0,$$

which is the line through the circumcentre O perpendicular to BC.

Similarly the lines EE, FF lead to the lines through O perpendicular to CA and AB respectively.

6. *Summary.*

The 27 points (pqr) may be classified in the following ways :—

I. 9 triads of collinear poles of the 9 triads DEF ; giving
   3 triads of concurrent lines ; giving
   1 triad of collinear points, viz. the Lemoine and Hessian points.

II. 9 triads of collinear points (viz. the lines DD) ; giving
   3 triads of concurrent lines ; giving
   1 triad of collinear points on the line through O $\perp^r$ to BC.

III. and IV. Similarly for EE and FF.

In addition to these triads we have

(a) 9 collinear triads AL, since every AL contains three points.

(b), (c) Similarly for BL, CL.

Since, in I., the Hessian axis passes through O, we have finally four lines I. to IV. passing through O, each line leading eventually by triads back to the 27 points (pqr).

It may be of interest to note that Ptolemy's Theorem may be used to find the equation of $D_{qr}\, D_{q+1,\, r+1}$, the coordinates of D_{qr} being written as

$$(\cos\pi/3,\ \cos\gamma_r,\ \cos\beta_q) \text{ or } \{\cos(-\pi/3),\ \cos\gamma_r,\ \cos\beta_q\}.$$

Also the parallelism of the lines EF, etc., will follow from the same theorem.

Morley's Trisection Theorem: an Extension and its Relation to the Circles of Apollonius.

By W. L. Marr, M.A., B.Sc.

(*Read 8th May 1914. Received 1st July 1914*).

Introductory Part.

1. In Morley's trisection theorem there are three triads of parallel lines which by their intersection with each other form equilateral triangles. The three lines EF, $E_{11}F_{11}$, $E_{22}F_{22}$ (*v.* Taylor and Marr) form one of these triads, and the equations are :—

$$(EF). \quad x\sin\alpha_2 + y\sin\beta_1 + z\sin\gamma_1 = 0 \quad \ldots\ldots\ldots\ldots(1)$$

$$(E_{11}F_{11}). \quad x\sin\alpha + y\sin\beta_2 + z\sin\gamma_2 = 0 \quad \ldots\ldots\ldots\ldots(2)$$

$$(E_{22}F_{22}). \quad x\sin\alpha_1 + y\sin\beta + z\sin\gamma = 0 \quad \ldots\ldots\ldots\ldots(3)$$

where $\alpha = \frac{1}{3}A$, $\alpha_1 = \frac{1}{3}A + \frac{2\pi}{3}$, $\alpha_2 = \frac{1}{3}A + \frac{4\pi}{3}$

and β, γ, β_1, γ_1, etc., are similarly defined.

Subtract (2) from (3) and

$$x\cos\alpha_2 + y\cos\beta_1 + z\cos\gamma_1 = 0 \quad \ldots\ldots\ldots\ldots(4)$$

Multiply (1) by $\cos\epsilon$ and (4) by $\sin\epsilon$, add, and

$$x\sin(\alpha_2 + \epsilon) + y\sin(\beta_1 + \epsilon) + z\sin(\gamma_1 + \epsilon) = 0 \quad \ldots\ldots\ldots\ldots(5)$$

(5) is a line parallel to (1), (2), or (3).

ϵ is arbitrarily chosen and temporarily fixed.

(5) will be named $\overline{211}_\epsilon$ or simply $\overline{211}$.

2. The line

$$x\sin(\alpha_2 + \epsilon) + y\sin(\beta_1 + \epsilon) + z\sin(\gamma_1 + \epsilon) = 0$$

or $\overline{211}_\epsilon$ is the polar of the point

$$[\sin(\alpha_2 + \epsilon),\ \sin(\beta_1 + \epsilon),\ \sin(\gamma_1 + \epsilon)]$$

or $(211)_\epsilon$ with respect to the imaginary conic $x^2 + y^2 + z^2 = 0$.

If the point (211) be given, the *corresponding* line $\overline{211}$ can be obtained by the following construction :—

(1) Find the isogonal conjugate of (211) and call it Q.

(2) Join AQ to meet BC in X′

BQ to meet CA in Y′, etc.

(3) Find X the harmonic conjugate of X′ with respect to BC.

So for Y and Z

The line XYZ is $\overline{211}$.

Similarly $\overline{121}$ is obtained from (121)

and $\overline{112}$ „ „ (112).

3. The point (211) lies on the line

$$x\sin(\beta-\gamma)+y\sin\left(\gamma-\alpha+\frac{4\pi}{3}\right)+z\sin\left(\alpha-\beta+\frac{2\pi}{3}\right)=0$$

or $$x\sin p+y\sin q_2+z\sin r_1=0$$

or briefly the line $\overline{pq_2r_1}$, where

$$p=\beta-\gamma,\ q=\gamma-\alpha,\ \text{etc.},\ p_1=p+\frac{2\pi}{3},\ \text{etc.}$$

(121) lies on the line $\overline{p_1qr_2}$,

(112) on $\overline{p_2q_1r}$.

The three lines $\overline{pq_2r_1}$, etc., were first discovered by Mr F. G. Taylor in another connexion and shown to meet in the symmedian point.

PART I.—*The First Group of Equilateral Triangles.*

4. To form an equilateral triangle three lines are required, viz., $\overline{211}_\epsilon$, $\overline{121}_\epsilon$, $\overline{112}_\epsilon$. This triangle would correspond in Morley's diagram, where $\epsilon=0$, to DEF.

In the same diagram

$$E_{11}F_{11} \text{ is } \overline{022}_{\epsilon=0}$$

$$E_{22}F_{22} \text{ is } \overline{100}_{\epsilon=0}.$$

5. The problem is to find $\overline{211_\epsilon}$, $\overline{121_\epsilon}$, $\overline{112_\epsilon}$. To find these lines a point $(111)_\epsilon$ is required. Call this point R. (*v.* § 6).

AR meets $\overline{pq_2r_1}$ in (211). [*v.* § (3).]

BR meets $\overline{p_1qr_2}$ in (121).

CR meets $\overline{p_2q_1r}$ in (112).

Hence by means of $(111)_\epsilon$ the three points $(211)_\epsilon$, etc., are found from which the equilateral triangle $\overline{211}$, $\overline{121}$, $\overline{112}$ can be derived. This triangle may be called ${}_\epsilon$DEF, and the point (111) the "primary point" from which it is derived. For ${}_\epsilon$DEF we may write simply DEF.

6. The point $(111)_\epsilon$ or, at length,

$$[\sin(\alpha_1+\epsilon), \sin(\beta_1+\epsilon), \sin(\gamma_1+\epsilon)]$$

lies on the line

$$x\sin p + y\sin q + z\sin r = 0,$$

or, briefly, $\overline{pqr}$, one of a group of three important lines $\overline{pqr}$, $\overline{p_1q_1r_1}$, $\overline{p_2q_2r_2}$ discovered by Mr Taylor and shown to meet in the point

$$\left[\sin\left(A+\frac{2\pi}{3}\right), \sin\left(B+\frac{2\pi}{3}\right), \sin\left(C+\frac{2\pi}{3}\right)\right],$$

or, briefly, $(\sin A_1 \sin B_1 \sin C_1)$ one of the points of intersection of the Apollonius circles. This point will be called H_1.

7. It will be assumed that ${}_\epsilon$DEF and ABC are in perspective. The axis of perspective is :—

$$\begin{vmatrix} x & y & z \\ 0 & \operatorname{cosec}(\beta_1+\epsilon) & -\operatorname{cosec}(\gamma_1+\epsilon) \\ -\operatorname{cosec}(\alpha_1+\epsilon) & 0 & \operatorname{cosec}(\gamma_1+\epsilon) \end{vmatrix} = 0$$

for ${}_\epsilon$EF meets BC in $[0, \operatorname{cosec}(\beta_1+\epsilon), -\operatorname{cosec}(\gamma_1+\epsilon)]$ and so for ${}_\epsilon$FD.

This equation becomes $\Sigma x\sin(\alpha_1+\epsilon)=0$, the equation of the the line $\overline{111}$, which *corresponds* to the primary point (111) from which the triangle DEF is derived. It will be assumed, without further proof, that the axis of perspective of any equilateral triangle, with ABC, is the line *corresponding* to the primary point from which that equilateral triangle is derived.

The line $\overline{111}_\epsilon$ passes through (pqr) for any value of ϵ since $\Sigma\sin(\alpha_1+\epsilon)\sin(\beta-\gamma)\equiv 0$. The characteristic property of ${}_\epsilon$DEF is that, for any value of ϵ, the axis of perspective with ABC passes through (pqr). In Morley's diagram, the axes of perspective of DEF, $D_{11}E_{11}F_{11}$, $D_{22}E_{22}F_{22}$ with ABC pass through (pqr).

The point (pqr) may be called the *source of perspective*, or, briefly, the *source* of ${}_\epsilon$DEF.

8. By operating from three primary points (000), (111), (222) on $\overline{pqr}$, we could reach the three triads of lines [$\overline{100}$, $\overline{010}$, $\overline{001}$], [$\overline{211}$, etc.], [$\overline{022}$, etc.], which, for a given value of ϵ, form a system of equilateral triangles corresponding to Morley's particular system where $\epsilon = 0$.

It will be better, however, to be content with (111) on $\overline{pqr}$ and proceed to $\overline{p_1 q_1 r_1}$ any point of which has coordinates of the type $\sin(\alpha_2 + \epsilon)$, $\sin(\beta_1 + \epsilon)$, $\sin(\gamma + \epsilon)$, or, briefly, $(210)_\epsilon$.

9. This point (210) will serve as the primary point of another type of equilateral triangle. Call (210) T.

Join AT to cut $\overline{p_1 q r_2}$ in (010) for the coordinates of a point on $\overline{p_1 q r_2}$ are at discretion (121), (202), or (010). (*v.* § 3).

Join BT to cut $\overline{p_2 q_1 r}$ in (220).

Join CT to cut $\overline{p q_2 r_1}$ in (211).

The three points (010), (220), (211) lead to the three *corresponding* lines from which an equilateral triangle of the type $D_{02}E_{21}F_{10}$ is formed.

It is interesting to note (i) that $\overline{211}$, a line which is related to A, is obtained from CT, that $\overline{010}$, which is related to B, is obtained from AT, and so for $\overline{200}$, (ii) that $D_{02}E_{21}$ is inclined at 120° to DE. We shall find later that $\overline{pqr}$ and $\overline{p_1 q_1 r_1}$, from primary points on which the triangles ϵDEF, $\epsilon D_{02}E_{21}F_{10}$ are derived, are also inclined at 120°.

10. The primary point being (210), the axis of perspective of $D_{02}E_{21}F_{10}$ with ABC is $\overline{210}$.

$\overline{210_\epsilon}$ passes through $(p_1 q_1 r_1)$ for any value of ϵ.

The characteristic of $\epsilon D_{02}E_{21}F_{10}$ is that its axis of perspective with ABC passes through $(p_1 q_1 r_1)$, which may be called the *source* of $D_{02}E_{21}F_{10}$. In Morley's particular system the axes of perspective of $D_{02}E_{21}F_{10}$, $D_{10}E_{02}F_{21}$, $D_{21}E_{10}F_{02}$ with ABC pass through $(p_1 q_1 r_1)$.

11. Proceed to $\overline{p_2 q_2 r_2}$, and take a point on it as primary point, viz. (012). From (012) we can reach the three lines $\overline{112}$, $\overline{022}$, $\overline{010}$ which form an equilateral triangle of the type $D_{01}E_{12}F_{20}$.

The axis of perspective is $\overline{012}$.

The *source* of $\epsilon D_{01}E_{12}F_{20}$ is $(p_2 q_2 r_2)$.

12. *Summary.*

By operating from primary points (111), (210), (012), we reach three types of triangles DEF, $D_{02}E_{21}F_{10}$, $D_{01}E_{12}F_{20}$, each with a characteristic *source*. These sources (pqr), $(p_1q_1r_1)$, $(p_2q_2r_2)$ are collinear on $\Sigma x \sin A_1 = 0$, which is interpreted as follows. The point $(\sin A_1, \sin B_1, \sin C_1)$ lies on $\overline{pqr}$, $\overline{p_1q_1r_1}$, $\overline{p_2q_2r_2}$, and may be regarded as $(111)_{\epsilon'}$, $(210)_{\epsilon''}$, or $(012)_{\epsilon'''}$. As (111) it will be the primary point of a triangle $_{\epsilon'}$DEF with source $\overline{p\,q\,r}$, as (210) it will be the primary point of another triangle $_{\epsilon''}D_{02}E_{21}F_{10}$ inclined at 120° to DEF with source $(p_1q_1r_1)$, as (012) it will be the primary point of $_{\epsilon'''}D_{01}E_{12}F_{20}$ with source $(p_2q_2r_2)$. But, as we are operating from $(\sin A_1, \sin B_1, \sin C_1)$, the axis of perspective of any of these three triangles with ABC is $\Sigma x \sin A_1 = 0$, on which, therefore, the three sources must lie. (*v.* § 7).

Part II.—*The Second Group of Equilateral Triangles.*

13. The primary points, from which the equilateral triangles can be derived, may be exhibited in three triads :—

(000), (111), (222) on $\overline{pqr}$
(210), (021), (102) on $\overline{p_1q_1r_1}$
(012), (120), (201) on $\overline{p_2q_2r_2}$.

The secondary points, which *correspond* to the sides of the equilateral triangles, are :—

(211), (022), (100) on $\overline{pq_2r_1}$
(121), (202), (010) on $\overline{p_1qr_2}$
(112), (220), (001) on $\overline{p_2q_1r}$.

The number of groups is 27, and there remain, therefore, three triads :—

(200), (011), (122) on $\overline{pq_1r_2}$
(020), (101), (212) on $\overline{p_2qr_1}$
(002), (110), (221) on $\overline{p_1q_2r}$.

These lines $\overline{pq_1r_2}$, etc., have been shewn by Mr Taylor to pass through the point $\left[\sin\left(A + \frac{4\pi}{3}\right), \text{etc.}\right]$, or, briefly, $(\sin A_2, \sin B_2, \sin C_2)$, the other point of intersection of the circles of Apollonius. This point will be called H_2.

It will be shewn later that, like the $\overline{p\,q\,r}$ lines, these $\overline{p\,q_1r_2}$ lines are inclined to each other at 120°, and that this property is reflected in the inclination to each other of the equilateral triangles to which these $\overline{p\,q_1r_2}$ lines are related.

14. As primary point take (200) on $\overline{p\,q_1r_2}$ and call it V.

AV meets $\overline{pq_2r_1}$ in (100).
BV meets $\overline{p_2q_1r}$ in (220).
CV meets $\overline{p_1qr_2}$ in (202).

The sides of the equilateral triangles derived from (200) are thus $\overline{100}$, $\overline{220}$, $\overline{202}$. This is the triangle $\epsilon D_{11}E_{10}F_{01}$. The triangle has to be read in the clockwise rotation. The transposition of E and F will be understood if it is noticed that AV meets its proper line $\overline{pq_2r_1}$, whereas BV meets $\overline{p_2q_1r}$, a line related to C and CV meets $\overline{p_1qr_2}$, a line related to B.

The primary point being (200), the axis of perspective with ABC is $\overline{200}$.

The *source* of $\epsilon D_{11}E_{10}F_{01}$ is (pq_1r_2).

In Morley's particular system the axes of perspective of $D_{11}E_{10}F_{01}$, $D_{22}E_{21}F_{12}$, and $D_{00}E_{02}F_{20}$ with ABC pass through (pq_1r_2).

15. Operating from (020) on $\overline{p_2qr_1}$ as primary point, we reach the triangle $\epsilon D_{01}E_{11}F_{10}$. The sides of this triangle are inclined at 120° to those of the previous, and $\overline{p_2qr_1}$, $\overline{pq_1r_2}$ are inclined at the same angle, as will be proved later.

The axis of perspective is $\overline{020}$, and the *source* is (p_2qr_1).

16. From (002) on $\overline{p_1q_2r}$ we reach $\epsilon D_{10}E_{01}F_{11}$. The *source* is (p_1q_2r).

17. The vertices of the nine equilateral triangles, if we take Morley's particular system, in this second group are distinct, but each vertex has already appeared as a vertex in the first group. The *sources* of the three types of triangles in the second group are collinear on $\Sigma x \sin A_2 = 0$, which is interpreted thus:

$(\sin A_2 \sin B_2 \sin C_2)$ lies on $\overline{pq_1r_2}$, $\overline{p_2qr_1}$, $\overline{p_1q_2r}$,

and may be regarded as $(200)_{\epsilon'}$ or $(020)_{\epsilon''}$ or $(002)_{\epsilon'''}$. It will thus give rise to three different equilateral triangles inclined to each other at 120°. As the axis of perspective with ABC of any of these is $\Sigma x \sin A_2 = 0$, the line which *corresponds* to the primary point ($\sin A_2$, etc.), it follows that the three *sources* must lie on $\Sigma x \sin A_2 = 0$ since each *source* lies on the axis of perspective.

PART III.—*The Geometry of* $\overline{100}_\epsilon$ *and the nine pqr lines.*

18. The line $\overline{100}_\epsilon$ or ${}_\epsilon E_{22} F_{22}$ is

$$x\sin(\alpha_1 + \epsilon) + y\sin(\beta + \epsilon) + z\sin(\gamma + \epsilon) = 0.$$

If $\overline{100}_\epsilon$ meet BC in K, the coordinates of K are $0, -\sin(\gamma + \epsilon), \sin(\beta + \epsilon)$. If K be found, $\overline{100}_\epsilon$ is a line through K inclined at $(\gamma - \beta)$ to CB. It will serve us better, however, to find X, where X is on BC and AX is the line isogonal to AK.

Construction for X. (Fig. 1).

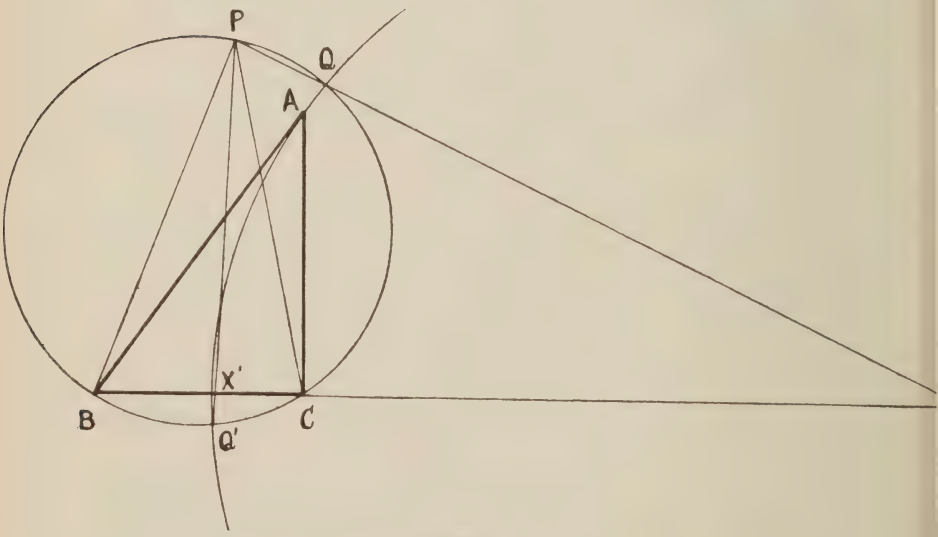

Fig. 1.

Draw the Apollonius circle through A. Make the angle $CBP = \beta + \epsilon$ and angle $BCP = \gamma + \epsilon$. Q and Q' are the intersections of the circle BPC and the Apollonius circle.

PQ meets BC in X. PQ′ meets BC in X′.

$$\frac{BX}{XC} = \frac{BX/BQ}{XC/CQ} \cdot \frac{BQ}{CQ} = \frac{\sin BQX}{\sin CQX} \cdot \frac{\sin C}{\sin B}$$

$$= \frac{\sin(\gamma+\epsilon)}{\sin(\beta+\epsilon)} \cdot \frac{\sin C}{\sin B}$$

(for $BQX = \pi - BQP = \pi - BCP = \pi - (\gamma+\epsilon)$

$CQX = PBC = \beta+\epsilon$,

and Q lies on the Apollonius circle,

$$\therefore\ BQ/CQ = \sin C/\sin B$$

$$\therefore\ BX\sin B/XC\sin C = \sin(\gamma+\epsilon)/\sin(\beta+\epsilon),$$

or if 0, y, z be the co-ordinates of X,

$$z/y = -\sin(\gamma+\epsilon)/\sin(\beta+\epsilon).$$

Similarly, if X′ be 0, y', z',

$$z'/y' = \sin(\gamma+\epsilon)/\sin(\beta+\epsilon),$$

and, therefore, (AB,XX′) is harmonic.

Find AK, the line isogonal to AX with K on BC. As the tangent at P is inclined at $(\gamma-\beta)$ to CB, draw through K the line parallel to this tangent, and $\overline{100_\epsilon}$ is obtained.

19. The line $\overline{p\,q\,r}$ is $\Sigma x\sin(\beta-\gamma)=0$. This line, as we have seen, is the locus of the primary points of the triangle $_\epsilon$DEF for any value of ϵ. In Morley's particular system Mr Taylor has shown that it is the "line of poles" of DEF, $D_{11}E_{11}F_{11}$, $D_{22}E_{22}F_{22}$. In the previous construction for $\overline{100_\epsilon}$ let $\epsilon = -\alpha$. (Fig. 2).

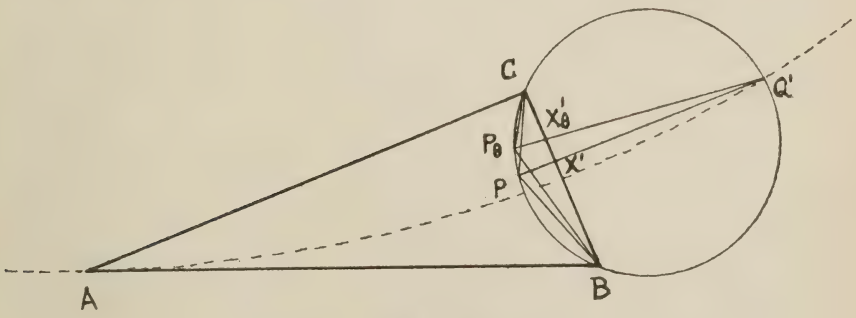

Fig. 2.

The angle $CBP = \beta - \alpha$ and $BCP = \gamma - \alpha$.

The angle $BQ'C = (\beta - \alpha + \gamma - \alpha) = \frac{\pi}{3} - A$.

Now Q′ lies on the Apollonius circle through A, and since $BQ'C = \frac{\pi}{3} - A$, it is (H_1) one of the two points of intersection of the three Apollonius circles, and its coordinates are $\sin A_1 \sin B_1 \sin C_1$. (*v.* § 6).

The coordinates of X′, the meet of PQ′ and BC, (*v.* § 18) are 0, $\sin(\beta - \alpha)$, $\sin(\gamma - \alpha)$ or 0, $-\sin(\alpha - \beta)$, $\sin(\gamma - \alpha)$, and hence X′ is a point on $\Sigma x \sin(\beta - \gamma) = 0$ or $\overline{p\,q\,r}$. But Q′ (or H_1) is also a point on $\overline{p\,q\,r}$.

$\therefore$ $\overline{pqr}$ is the line Q′X′.

20. The line $(\overline{pqr})_\theta$ is $\Sigma x \sin(\beta - \gamma + \theta) = 0$. Rotate the CP of Fig. 2 to the right through an angle θ to the position CP_θ.

The angle $CBP_\theta = \beta - \alpha - \theta$,

and the angle $BCP_\theta = \gamma - \alpha + \theta$.

P_θ lies on the circle BPC which intersects the Apollonius circle in Q′ or H_1.

Join $P_\theta Q'$ to meet BC in X'_θ.

The coordinates of X'_θ (*v.* § 18) are

$$0,\ \sin(\beta - \alpha - \theta),\ \sin(\gamma - \alpha + \theta),$$

or

$$0,\ -\sin(\alpha - \beta + \theta),\ \sin(\gamma - \alpha + \theta),$$

and, therefore, X'_θ is a point on $(\overline{pqr})_\theta$.

Since $\Sigma \sin A_1 \sin(\beta - \gamma + \theta) \equiv 0$, Q′ (or H_1) is also a point on $(\overline{pqr})_\theta$ and hence $Q'X'_\theta$ is the line $(\overline{pqr})_\theta$.

21. The angle between Q′X′ and $Q'X'_\theta$ is $PQ'P_\theta = PCP_\theta = \theta$, Thus $(\overline{pqr})_\theta$ is inclined at an angle θ to $\overline{pqr}$. When $\theta = \frac{2\pi}{3}$, $(\overline{pqr})_\theta$ becomes $\overline{p_1 q_1 r_1}$, and hence $\overline{p_1 q_1 r_1}$ and $\overline{pqr}$ are inclined to each other at 120°, and so for $\overline{p_2 q_2 r_2}$ and $\overline{p_1 q_1 r_1}$.

22. Any line through H_1 is of the form $\Sigma x\sin(\beta-\gamma+\theta)=0$. When $\theta=\gamma-\beta$ this equation becomes

$$y\sin(2\gamma-\alpha-\beta)+z\sin(\gamma+\alpha-2\beta)=0,$$

or
$$y\sin\left(C-\frac{\pi}{3}\right)+z\sin\left(\frac{\pi}{3}-B\right)=0,$$

or
$$y\sin C_1-z\sin B_1=0.$$

This is the line AH_1, and thus AH_1 is inclined at $(\gamma-\beta)$ to $\overline{pqr}$.

23. The line $\overline{p\,q_1r_2}$ is

$$x\sin(\beta-\gamma)+y\sin\left(\gamma-\alpha+\frac{2\pi}{3}\right)+z\sin\left(\alpha-\beta+\frac{4\pi}{3}\right)=0.$$

This line is the locus of the primary points of the triangle $\epsilon D_{11}E_{10}F_{01}$. In Morley's particular system Mr Taylor has shown that it is the "line of poles" of $D_{11}E_{10}F_{01}$, $D_{22}E_{21}F_{12}$, $DE_{02}F_{20}$.

In the construction for $\overline{100}_\epsilon$ (*v.* §18) let

$$\epsilon=\frac{2\pi}{3}-\alpha \quad \text{(Fig. 3).}$$

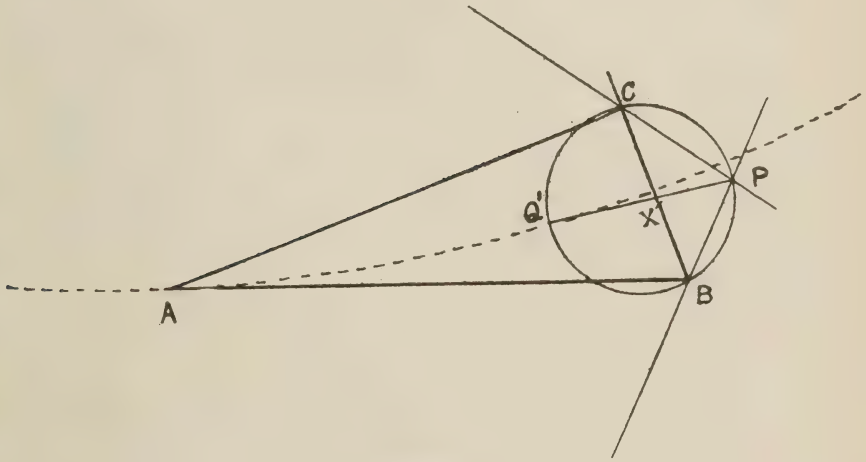

Fig. 3.

The angle

$$BPC=\left(\frac{2\pi}{3}+\beta-\alpha\right)+\left(\frac{2\pi}{3}+\gamma-\alpha\right)-\pi=\frac{2\pi}{3}-A.$$

The circle BPC cuts the Apollonius circle in Q′ on the side of BC remote from P.

The angle

$$BQ'C = \pi - \left(\frac{2\pi}{3} - A\right) = \frac{\pi}{3} + A.$$

Hence Q′ is the second point of intersection of the three Apollonius circles, viz. H_2, and its coordinates are $\sin A_2$, $\sin B_2$, $\sin C_2$. (*v.* § 13).

If X′ is the meet of PQ′ and BC, QX′ is the line $\overline{(p\,q_1r_2)}$. (*cf.* § 19).

24. The line $\overline{(p\,q_1r_2)_\theta}$ is

$$x\sin(\beta - \gamma + \theta) + y\sin\left(\gamma - \alpha + \frac{2\pi}{3} + \theta\right) + z\sin\left(\alpha - \beta + \frac{4\pi}{3} + \theta\right) = 0.$$

By a proof similar to that of § 21 it can be shown that $\overline{(p\,q_1r_2)_\theta}$ and $\overline{p\,q_1r_2}$ are inclined at an angle θ.

Hence (*cf.* § 21) $\overline{pq_1r_2}$, $\overline{p_1q_2r}$, $\overline{p_2qr_1}$ are inclined to each other at 120°.

25. AH_2 is inclined to $\overline{p\,q_1r_2}$ at $(\gamma - \beta)$ (*cf.* § 22).

26. As AH_1 is inclined at $(\gamma - \beta)$ to $\overline{p\,q\,r}$ and AH_2 has the same inclination to $\overline{p\,q_1r_2}$, it follows that $\overline{p\,q\,r}$ and $\overline{p\,q_1r_2}$ make with each other an angle equal to H_1AH_2.

Hence $\overline{p\,q\,r}$ and $\overline{p\,q_1r_2}$ intersect on the circumference of the Apollonius circle through A.

$\overline{pqr}$ and $\overline{pq_1r_2}$ meet on the Apollonius circle (A).
$\overline{p_1q_1r_1}$ and $\overline{p_1q_2r}$,, ,, ,, ,,
$\overline{p_2q_2r_2}$ and $\overline{p_2qr_1}$,, ,, ,, ,,
$\overline{pqr}$ and $\overline{p_2qr_1}$, etc., meet on the Apollonius circle (B).
$\overline{pqr}$ and $\overline{p_1q_2r}$, ,, ,, ,, ,, ,, (C).

(*v.* Fig. 5.)

27. The line $\overline{pq_2r_1}$ is

$$x\sin(\beta - \gamma) + y\sin\left(\gamma - \alpha + \frac{4\pi}{3}\right) + z\sin\left(\alpha - \beta + \frac{2\pi}{3}\right) = 0.$$

This line is the locus of the points $(211)_\epsilon$ which *correspond* to the lines $\overline{211}_\epsilon$ or $_\epsilon$EF. In Morley's particular system Mr Taylor has shown that it is the "line of poles" of a certain triad of non-equilateral triangles.

In the construction for $\overline{100}_\epsilon$ (*v.* § 18) let

$$\epsilon = \frac{4\pi}{3} - \alpha \quad \text{(Fig. 4).}$$

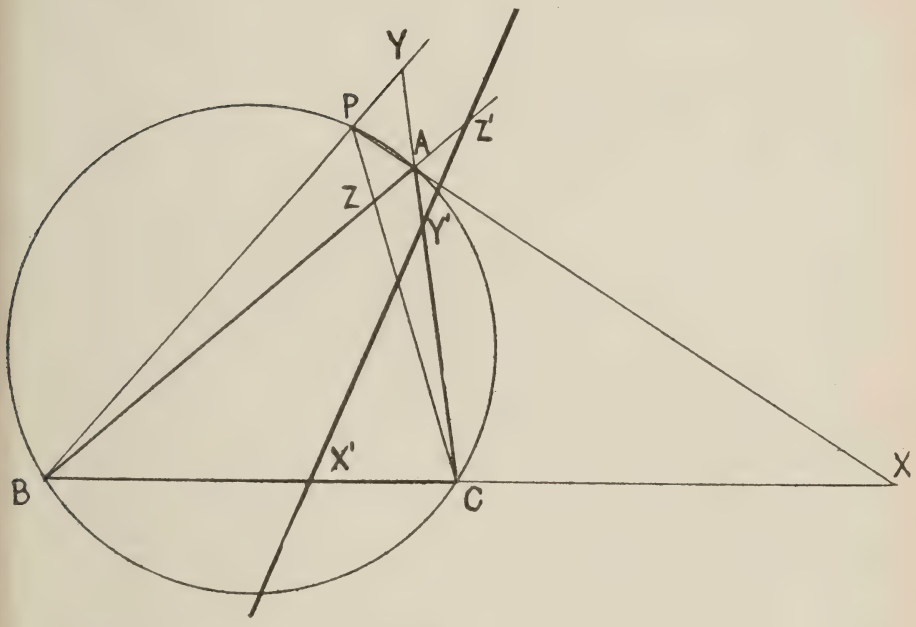

Fig. 4.

The angle

$$\text{BPC} = \pi - \left[\left(\frac{4\pi}{3} + \beta - \alpha\right) + \left(\frac{4\pi}{3} + \gamma - \alpha\right) - 2\pi\right] = \text{A}.$$

Hence P lies on the circumcircle of ABC and the circle BPC or ABC meets the Apollonius circle in A, and the tangent at P is parallel to EF.

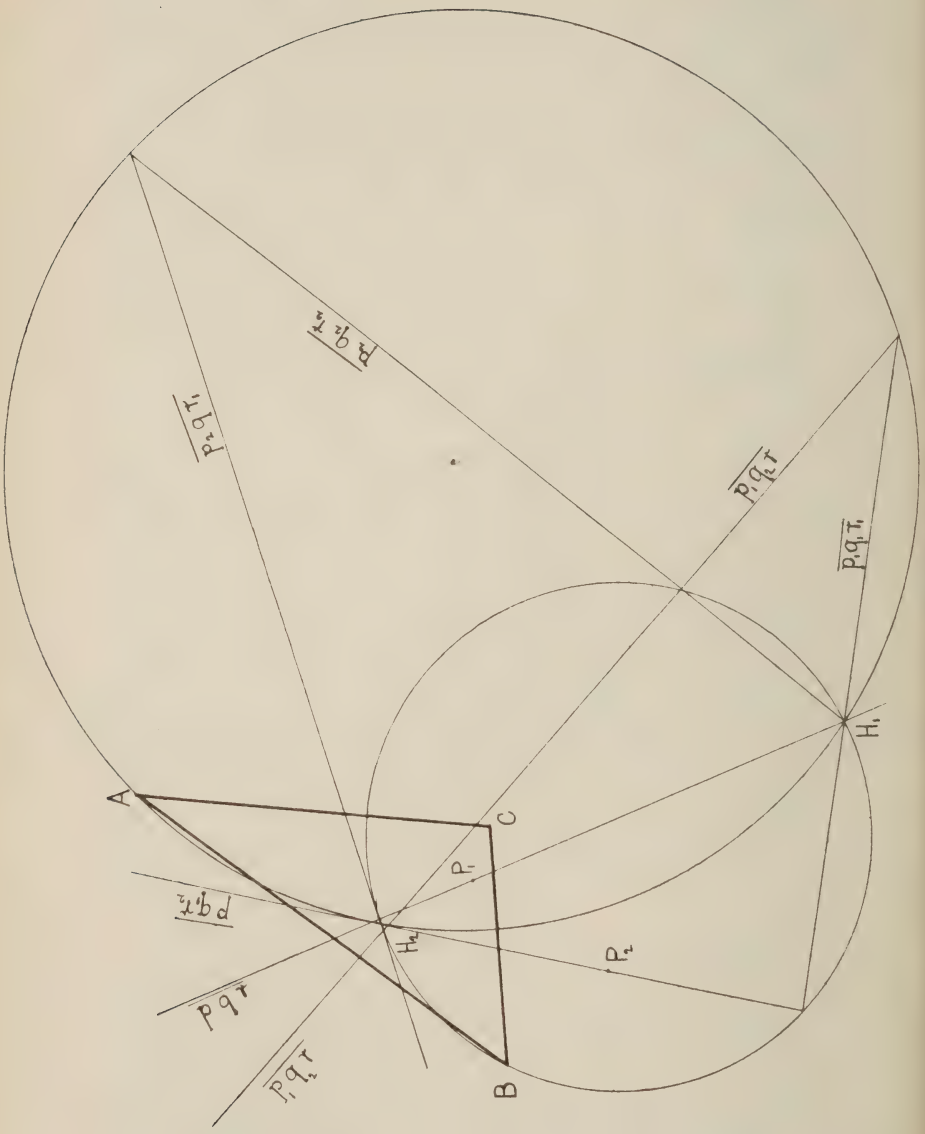

Fig. 5.

A now takes the place of Q in the construction for $\overline{100_\epsilon}$ and the coordinates of X, the point of intersection of PA and BC, are (*v.* § 18)

$$0, \quad -\sin\left(\frac{4\pi}{3}+\beta-\alpha\right), \quad \sin\left(\frac{4\pi}{3}+\gamma-\alpha\right),$$

or

$$0, \quad \sin\left(\alpha-\beta+\frac{2\pi}{3}\right), \quad \sin\left(\gamma-\alpha+\frac{4\pi}{3}\right).$$

X, therefore, is the harmonic conjugate of the meet of $\overline{p\,q_2r_1}$ and BC.

Similarly Y is the harmonic conjugate of the meet of $\overline{p\,q_2r_1}$ and CA. So for Z.

The construction for $\overline{p\,q_2r_1}$ is, then, as follows:— (*v.* Fig. 4).

Find P the point of contact of the upper tangent to the circle ABC, which is parallel to EF. Let PA meet BC in X, etc. Let X′ be the harmonic conjugate of X with regard to BC, etc.

The line X′Y′Z′ is $\overline{pq_2r_1}$.

The three lines $\overline{p\,q_2r_1}$, $\overline{p_1q\,r_2}$, $\overline{p_2q_1r}$ are obtained in the same manner from three points P, which lie on the circumference of ABC, and are the vertices of an equilateral triangle with sides parallel to those of DEF.

28. Let the coordinates of P on the circumcircle be $1/\alpha$, $1/\beta$, $1/\gamma$. Since the circumcircle is the isogonal conjugate of the line at infinity, (α, β, γ) is a point on the line at infinity, and, therefore, $\alpha\sin A+\beta\sin B+\gamma\sin C=0$. The line X′Y′Z′ can easily be proved to have the equation $x\alpha+y\beta+z\gamma=0$, and therefore must pass through the symmedian point (sinA, sinB, sinC) since

$$x:y:z=\sin A:\sin B:\sin C$$

satisfies $\Sigma x\alpha=0$.

Take any point on X′Y′Z′, the coordinates of which are $(211)_\epsilon$, but which may now be named $x_0y_0z_0$. Just as the point (211) *corresponds* to the line $\overline{211}$, so the point $(x_0y_0z_0)$ corresponds to

$$xx_0+yy_0+zz_0=0.$$

But since $(x_0y_0z_0)$ is on $\Sigma x\alpha=0$,

$$x_0\alpha+y_0\beta+z_0\gamma=0,$$

and therefore the equation $\Sigma xx_0=0$ is satisfied by

$$x:y:z=\alpha:\beta:\gamma.$$

This means that $\Sigma x x_0 = 0$ always passes through $(\alpha\beta\gamma)$ for any position of $x_0 y_0 z_0$ on X′Y′Z′. Since $(\alpha\beta\gamma)$ is a point at infinity, it follows that $\Sigma x x_0 = 0$ remains parallel for different positions of $x_0 y_0 z_0$. It has already been shewn that, for any value of ϵ, $\overline{211_\epsilon}$ is parallel to EF, but this proof, which is independent of the particular position of P on the circumcircle, shows that a series of parallel lines related to P can be obtained for any position of P on the circumcircle.

29. If the coordinates of P be $\operatorname{cosec}(B-C)$, $\operatorname{cosec}(C-A)$, $\operatorname{cosec}(A-B)$, then the X′Y′Z′ corresponding to this P is $\Sigma x \sin(B-C) = 0$. This line is the Hessian axis, and contains the two points $H_1(\sin A_1$, etc.) and $H_2(\sin A_2$, etc.). Hence (§ 28) $\Sigma x \sin A_1 = 0$ and $\Sigma x \sin A_2 = 0$ are parallel, *i.e.* the two lines which contain respectively the *sources* of the triangles of the first and second groups are parallel.

I wish to thank Mr F. Glanville Taylor and the Editors for helpful criticism and suggestions.

JOHN STURGEON MACKAY, M.A., LL.D.

Dr John S. Mackay, the first President of our Society, died on the 25th of March, 1914, at his residence, 69 Northumberland Street, Edinburgh. In view of the unrivalled services he rendered the Society, and the active interest he took in its work from its inception, it has been judged fitting that the *Proceedings* should contain a short record of his life.

John Sturgeon Mackay was of Highland descent by his father's side, but of Lowland descent by his mother's, and it was during a visit of his mother to her relatives at Auchencairn, near Kirkcudbright, that he was born on the 22nd of October 1843. It was in Perth, however, he spent his boyhood and attended school. Fortunate in growing up in a city situated amid delightful scenery and associated with many stirring events of Scottish history, he was not less happy in the school where he was educated. Dr Miller, the rector of Perth Academy from 1837 to 1881, was a remarkable man; in the words of one of his distinguished pupils, "under his reign the Academy was less a school than a notable provincial college." The obituary notice of Dr Miller, contributed by Dr Mackay to the *Proceedings of the Royal Society of Edinburgh*, Vol. XIX., Session 1891–92, presents a striking picture of the finest type of the old Scottish schoolmaster, and outlines in terse but graphic language the relations between the master and the pupil. From that notice I extract the following statement of the course of mathematics through which Dr Miller conducted his junior and senior classes:—"The course consisted of the theory of arithmetic, algebra, plane and solid geometry,

geometrical conics, plane and spherical trigonometry, dynamics, elementary physics and astronomy, and inorganic chemistry. Occasionally he had pupils who gave a third or even a fourth year's attendance, for which he would accept no fee, and he took particular pleasure in initiating them into the mysteries of the calculus." One might be tempted to hint that such a wide range of subjects would be only superficially covered, but I know, from frequent conversations with Dr Mackay, that in his case at least the work was very thoroughly done; to the end of his days he could repeat *verbatim* many of the enunciations in geometry and in spherical trigonometry as he had learned them under Dr Miller.

The rector was a really competent mathematician; but "his culture was not that of science alone, for he was widely read in literature, and he could grapple with the philosophical and theological questions of the day." An Academy boy had excellent opportunities for a sound training in English, Latin, and Greek, as well as in Mathematics and Science, and when Mackay matriculated at St Andrews in 1859, he was admirably prepared for entering on University studies.

During the years in which he was a student at St Andrews the classes were small, and the students were bound together in a cameraderie that is impossible in a large University. At that period the curriculum for the M.A. degree was fixed, and it was usual for the students who entered the University in the same year to keep together and attend the same lectures throughout the course. Dr Mackay was fond of recalling incidents of his undergraduate days, and he could trace the subsequent career of nearly every member of his own class. He had as fellow-students several men who afterwards rose to eminence; of them all, the one for whom he had the greatest admiration was the late Rev. Dr Crichton of Annan. He considered Dr Crichton to be one of the finest scholars St Andrews ever sent forth.

Mackay was soon recognised as one of the "good students." He took kindly to every subject included in the degree course, and was in the honours list of every class. He read widely in

general literature, and he did not confine his technical studies to prescribed books and authors. The stores of learning he then amassed were a permanent possession, and there are few graduates who retain through life as ready a command as he did of the learning acquired at the University.

He completed his attendance at the classes, and passed all the examinations for the M.A. Degree in 1863. He did not graduate, however, till 1865, as he wished to take the examinations for an Honours Degree, but he was prevented by various causes from realising his wish. He had thoughts of going forward to the ministry of the United Presbyterian Church, and as at that time the theological classes met for two months only in the autumn of each year, he followed a usual custom of taking a teaching engagement, and he found congenial occupation as assistant to Dr Miller in Perth Academy. He attended the Theological Hall for three sessions, but he gradually made up his mind to devote himself to teaching as a profession, and after an apprenticeship of three years under Dr Miller, he obtained an appointment on the staff of the Edinburgh Academy. From 1866 till his retiral in 1904, he remained at the Academy, for the greater part of that period as head mathematical master.

During his early years in Edinburgh he spent a considerable part of his leisure in the study of botany and geology, and he attended several University classes, taking part in the various class excursions. He retained his interest in these subjects to the last, and he often expressed to me his belief in the value of botany as a subject of general education. His knowledge of botany was, for an amateur's, remarkably accurate, and, like all his knowledge, was always at hand.

In the early seventies he had a long holiday in France, and from that date to the close of his life, there were few years in which he did not spend the greater part of his vacation on the Continent, and chiefly in Paris. He gradually acquired a thorough mastery of French, both literary and colloquial, and read very widely in French literature; he made many friends in Paris, and indeed seemed to feel quite as much at home

there as in Edinburgh. Though he did not obtain the same proficiency in German and Italian as in French, he had a really good knowledge of these languages, and when one was asked to recommend him a book, it made very little difference to him whether the language in which it was composed was English, French, German or Italian. He had, besides, a scholarly knowledge of Latin and Greek, so that when he had fairly entered on historical studies he had at command a linguistic equipment such as professed historians do not always possess.

In Dr Mackay's notice of his old master occurs a sentence that might have been written of himself:—"While he was interested in science mainly for its own sake and for the sake of the benefits which its discoveries conferred on mankind, he was keenly alive to the educational importance of its historical development, and familiarised his pupils with the names and the achievements of the great masters from Euclid downwards." It is not unreasonable to conclude that Dr Mackay's interest in the history of mathematics was awakened in his school days; that interest was certainly of life-long duration. From an early period his attention was directed to the Greek geometers, and one of the disappointments of his life is associated with these earlier studies. The incident is so graphically described in a notice of Dr Mackay that appears in the May number of *The Edinburgh Academy Chronicle* over the initials D. W. T., that I cannot refrain from quoting the whole passage:—

"Of all the great Alexandrine geometers, the works of one only, Pappus, remained to be edited. The Oxford scholars of the eighteenth century had neglected him, and a sixteenth-century Latin translation, together with a bare reprint of a small part of his text by a French editor, was all that was extant. For years Mackay had given his holidays to the collation of manuscripts, and his nights to the patient interpretation of the old mathematician; and at last his work was done, and the whole book lay complete in his desk, in his beautiful handwriting, with a wealth of drawings. He went in one day to Williams and Norgate's foreign bookshop in

Frederick Street, then managed by Mr Wheatley, an excellent scholar; and Mr Wheatley said, 'I've something to-day, Mr Mackay, that will interest you,' and he produced the first volume of Hultsch's edition of Pappus, a work that has held the field to this day, and is the *magnum opus* of its distinguished author. Mackay took the book home. He found that Hultsch and he had collated the same MSS., had arrived at the same interpretations, had noted the same difficulties, down to the smallest—a wrong letter, an omitted κάι—the German had noted them all. Some men, in Mackay's place, would have made all haste to send their own book to the printer, hoping to gain, by all but contemporaneous publication, some part of the scholar's reward. But Mackay had a sterner code of duty and of honour. The work was done, and well done; the needs of his few fellow-students of Greek mathematics were sufficiently met; and what was lost was a matter for him alone. A few years later, in a less direct and poignant way, Mackay found himself forestalled by the publication of Allman's *Greek Geometry*. This was a book that Mackay could have written to perfection, and perhaps with even greater learning than the Irish scholar displayed."

The head mathematical master of the Academy had gradually become known in educational circles as a scholar and as a mathematician, and when, at a meeting held in the Mathematical Class Room of Edinburgh University on the 2nd of February 1883, it was agreed to institute a Society to be called "The Edinburgh Mathematical Society," his unique position was recognised in his appointment as its first President. From that day till the close of his life he was unwearied in his efforts to promote the welfare of the Society. Until failing health confined him to the house in the evening, he was regular in his attendance, and by his kindness and courtesy in welcoming new members, he greatly contributed to that friendly atmosphere which distinguished the meetings. The Society had a hard struggle in its early days. It first Secretary, Dr Knott, was appointed Professor of Physics at Tokio soon after the Society was founded. In the second session the late

Mr A. Y. Fraser was elected as Dr Knott's successor in the Secretaryship, and he held the office till the end of the sixth session. During these critical years Dr Mackay and Mr Fraser were in constant communication about the affairs of the Society, and it is in no small degree due to their devoted service that the early difficulties were overcome and the new organisation fairly launched on its career.

A glance at the index of the first twenty volumes of the *Proceedings*, drawn up with consummate skill by Dr Mackay himself, shows how regular were his contributions. Most of his articles are geometrical, and they always include historical notes and references. They represent in every instance the results of his own investigation, and I think I am correct in stating that in no single case has his accuracy been impugned. The first paper he submitted is an admirable example of his characteristic style—precision of language, lucidity of arrangement, and accuracy of citation. In the course of a long series of articles he throws a flood of light on the origin of many geometrical theorems, traces their developments, indicates their relationships, and often gives to them their finishing touch. He settled the source of the misnamed Simson Line, and in one of his last papers he made important additions to the bibliography of the envelope of that line. The number of references in his articles, the obscure journals and books that furnished valuable results, the variety of languages from which the facts are cited, the scrupulous accuracy of quotation and reference, can best be understood by those who have themselves attempted historical research. There is no parade of learning, but the fruit of long and laborious first-hand investigation is evident on every page.

There is one field of investigation which Dr Mackay had made peculiarly his own ; it is what he himself called English "mathematical journalism" as distinct from the Proceedings and Transactions of learned Societies. No one who desires to become acquainted with what may be called the more homely forms of English mathematical research can find a better—or rather, any other—guide than his article *Notice sur le*

Journalisme Mathématique en Angleterre, contributed to the French Association for the Advancement of Science (Congrès de Besançon, 1893). The preparation of that article was in itself a remarkable feat. One of the members who had promised a paper was unable to be present, and on the afternoon of the day before the paper was set down for reading Dr Mackay was asked to take the absent member's place. With his usual courtesy, he agreed to try "to put some things together"; the result was the notice in almost the exact form in which it appears in the volume of the Association for that year.

Although Dr Mackay's contributions to the *Proceedings* of our Society are mainly geometrical, it is proper to note that the Report on the Teaching of Arithmetic in Vol. VI. owes very much to him, and many of the articles are from his pen.

In recognition of the great services he had rendered the Society, he was elected an Honorary Member in November 1894.

Of the textbooks written by Dr Mackay, the most characteristic was his *Elements of Euclid*, Books I to VI, published in 1884, and enlarged in 1888 by the addition of the usual propositions from Books XI and XII. Many of its distinctive features, especially its historical references, were soon imitated by other writers; but there are many subtle deviations from Simson's text that escape the notice of the casual reader, and that lend a unity to several propositions that is usually ignored. The composition of this book was a labour of love; much of the original manuscript had to be sacrificed to the exigencies of publication; but as a textbook of general education it is, in my opinion, the best adaptation of Euclid's Elements that we possess. When in the early years of the present century the demand for a new type of textbook arose, he felt constrained to comply with it, and he produced his *Plane Geometry* (Books I, II, III, in 1904; Books IV, V, in 1905). The new book is an excellent piece of work, but he did not fully sympathise with the new movement, and his attitude is, I think, reflected in this later book.

In 1869 he published a volume of Arithmetical Exercises, and in 1899 a formal textbook with the title *Arithmetic, Theoretical and Practical.* (The historical bent of the writer shows itself at every convenient opportunity, and a careful reader may learn some curious facts).

Of Dr Mackay's other activities I shall only make brief mention. He was a member of the A.I.G.T. from its origin, and took an active interest in its work. A Fellow of the Royal Society of Edinburgh, he served for two periods on the Council, and his services were specially valuable on the Library Committee. He frequently attended the meetings of the French Association for the Advancement of Science, and corresponded regularly with several well-known French mathematicians. He was a member of the Permanent International Commission for Mathematical Bibliography, and he was of special service through his great linguistic attainments. He took a warm interest in the Franco-Scottish Society, and was a member both of the Council and of the Executive.

In 1887 his old University recognised his great learning by conferring on him the Honorary Degree of LL.D.

In 1904 he retired from his position in the Academy. The number of the *Academy Chronicle* from which I have already quoted contains a graceful tribute to his memory from two of his pupils, and those who wish to understand the relations that existed between Dr Mackay and generations of boys should read these characteristic appreciations of their old master. Like many bachelors, he was fond of young people, and speedily gained their confidence; no master was ever prouder of the successes of his pupils or took a livelier interest in their after welfare; the visits he received from those who had gone out into the world and risen to eminence were honourable to master and pupil alike.

In his youth Dr Mackay was an enthusiastic golfer. During one of his sessions at St Andrews he and a fellow-student determined to make one round of the links, if it were possible, every day; they succeeded on all but three days, when the weather proved too much even for their exuberant spirits, and

compelled them to desist before they had well begun. In later years he became a member of a bowling-green, and acquired considerable skill at the game, but the sport in which he specially excelled was skating. He hoped that after his retiral he would be able to resume his golf, but he had, through want of practice, lost much of his youthful skill, and he seldom appeared on the links. His daily walk provided him with such physical exercise as he found it necessary to take.

He was a delightful companion, blessed with the saving grace of humour, and those who had the privilege of his friendship will never forget the genial talk in which he ranged over the most varied fields of knowledge, passing lightly from the gossip of the day to the discussion of the deepest problems of philosophy. His culture was both literary and scientific, his scholarship was exact, but free from pedantry, and his vast stores of knowledge were always at hand, ready for instant use. Of a genial and kindly disposition, he was ever ready to offer a helping hand to his less fortunate fellowmen. Charitable in his judgments of others, he kept a high standard of ethics for himself, and was his own severest critic.

In the beginning of this year the approach of serious trouble was indicated by the rapid failure of his eyesight; he became very ill in the first days of March, and after a short rally passed away on the afternoon of the 25th of that month. On Saturday, the 28th, he was laid to rest in the city where he spent his boyhood, and where he laid the foundations of that learning for which in after life he was so distinguished.

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